

Chapter 4

Partial Fractions

4.1 Introduction: A fraction is a symbol indicating the division of integers. For example, $\frac{13}{9}$, $\frac{2}{3}$ are fractions and are called Common Fraction. The dividend (upper number) is called the numerator $N(x)$ and the divisor (lower number) is called the denominator, $D(x)$.

From the previous study of elementary algebra we have learnt how the sum of different fractions can be found by taking L.C.M. and then add all the fractions. For example

$$\begin{aligned} \text{i) } & \frac{1}{x-1} + \frac{2}{x+2} = \frac{3x}{(x-1)(x+2)} \\ \text{ii) } & \frac{2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x-2} = \frac{9x^2+5x-3}{(x+1)^2(x-2)} \end{aligned}$$

Here we study the reverse process, i.e., we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called **Partial fractions**.

4.2 Partial fractions :

To express a single rational fraction into the sum of two or more single rational fractions is called **Partial fraction resolution**.

For example,

$$\frac{2x + x^2 - 1}{x(x^2 - 1)} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$$

$$\frac{2x + x^2 - 1}{x(x^2 - 1)} \text{ is the resultant fraction and } \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1} \text{ are its}$$

partial fractions.

4.3 Polynomial:

Any expression of the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ where $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are real constants, if $a_n \neq 0$ then $P(x)$ is called polynomial of degree n .

4.4 Rational fraction:

We know that $\frac{p}{q}$, $q \neq 0$ is called a rational number. Similarly

the quotient of two polynomials $\frac{N(x)}{D(x)}$ where $D(x) \neq 0$, with no common factors, is called a rational fraction. A rational fraction is of two types:

4.5 Proper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called a proper fraction if the degree of numerator $N(x)$ is less than the degree of Denominator $D(x)$.

For example

$$(i) \quad \frac{9x^2 - 9x + 6}{(x-1)(2x-1)(x+2)}$$

$$(ii) \quad \frac{6x + 27}{3x^3 - 9x}$$

4.6 Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called an improper fraction if the degree of the Numerator $N(x)$ is greater than or equal to the degree of the Denominator $D(x)$

For example

$$(i) \quad \frac{2x^3 - 5x^2 - 3x - 10}{x^2 - 1}$$

$$(ii) \quad \frac{6x^3 - 5x^2 - 7}{3x^2 - 2x - 1}$$

Note: An improper fraction can be expressed, by division, as the sum of a polynomial and a proper fraction.

For example:

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Which is obtained as, divide $6x^3 + 5x^2 - 7$ by $3x^2 - 2x - 1$ then we get a polynomial $(2x+3)$ and a proper fraction $\frac{8x - 4}{x^2 - 2x - 1}$

4.7 Process of Finding Partial Fraction:

A proper fraction $\frac{N(x)}{D(x)}$ can be resolved into partial fractions as:

- (I) If in the denominator $D(x)$ a linear factor $(ax + b)$ occurs and is non-repeating, its partial fraction will be of the form

$$\frac{A}{ax + b}, \text{ where } A \text{ is a constant whose value is to be determined.}$$

- (II) If in the denominator $D(x)$ a linear factor $(ax + b)$ occurs n times, i.e., $(ax + b)^n$, then there will be n partial fractions of the form

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \dots + \frac{A_n}{(ax + b)^n}$$

,where $A_1, A_2, A_3, \dots, A_n$ are constants whose values are to be determined

- (III) If in the denominator $D(x)$ a quadratic factor $ax^2 + bx + c$ occurs and is non-repeating, its partial fraction will be of the form

$$\frac{Ax + B}{ax^2 + bx + c}, \text{ where } A \text{ and } B \text{ are constants whose values are to be determined.}$$

- (IV) If in the denominator a quadratic factor $ax^2 + bx + c$ occurs n times, i.e., $(ax^2 + bx + c)^n$, then there will be n partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3x + B_3}{(ax^2 + bx + c)^3} + \dots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

Where $A_1, A_2, A_3, \dots, A_n$ and $B_1, B_2, B_3, \dots, B_n$ are constants whose values are to be determined.

Note: The evaluation of the coefficients of the partial fractions is based on the following theorem:

If two polynomials are equal for all values of the variables, then the coefficients having same degree on both sides are equal, for example, if

$$px^2 + qx + a = 2x^2 - 3x + 5 \quad \forall x, \text{ then}$$

$$p = 2, \quad q = -3 \text{ and } a = 5.$$

4.8 Type I

When the factors of the denominator are all linear and distinct i.e., non-repeating.

Example 1:

Resolve $\frac{7x - 25}{(x - 3)(x - 4)}$ into partial fractions.

Solution:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{A}{x - 3} + \frac{B}{x - 4} \text{-----(1)}$$

Multiplying both sides by L.C.M. i.e., $(x - 3)(x - 4)$, we get

$$7x - 25 = A(x - 4) + B(x - 3) \text{----- (2)}$$

$$7x - 25 = Ax - 4A + Bx - 3B$$

$$7x - 25 = Ax + Bx - 4A - 3B$$

$$7x - 25 = (A + B)x - 4A - 3B$$

Comparing the co-efficients of like powers of x on both sides, we have

$$7 = A + B \text{ and}$$

$$-25 = -4A - 3B$$

Solving these equation we get

$$A = 4 \text{ and } B = 3$$

Hence the required partial fractions are:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x - 3} + \frac{3}{x - 4}$$

Alternative Method:

Since $7x - 25 = A(x - 4) + B(x - 3)$

Put $x - 4 = 0, \Rightarrow x = 4$ in equation (2)

$$7(4) - 25 = A(4 - 4) + B(4 - 3)$$

$$28 - 25 = 0 + B(1)$$

$$B = 3$$

Put $x - 3 = 0 \Rightarrow x = 3$ in equation (2)

$$7(3) - 25 = A(3 - 4) + B(3 - 3)$$

$$21 - 25 = A(-1) + 0$$

$$-4 = -A$$

$$A = 4$$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x - 3} + \frac{3}{x - 4}$$

Note : The R.H.S of equation (1) is the identity equation of L.H.S

Example 2:

write the identity equation of $\frac{7x - 25}{(x - 3)(x - 4)}$

Solution : The identity equation of $\frac{7x - 25}{(x - 3)(x - 4)}$ is

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{A}{x - 3} + \frac{B}{x - 4}$$

Example 3:

Resolve into partial fraction: $\frac{1}{x^2 - 1}$

Solios: $\frac{1}{x^2 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1}$

$$1 = A(x + 1) + B(x - 1) \quad (1)$$

Put $x - 1 = 0$, $\Rightarrow x = 1$ in equation (1)

$$1 = A(1 + 1) + B(1 - 1) \quad \Rightarrow \quad A = \frac{1}{2}$$

Put $x + 1 = 0$, $\Rightarrow x = -1$ in equation (1)

$$1 = A(-1 + 1) + B(-1 - 1)$$

$$1 = -2B, \quad \Rightarrow \quad B = \frac{1}{2}$$

$$\frac{1}{x^2 - 1} = \frac{1}{2(x - 1)} - \frac{1}{2(x + 1)}$$

Example 4:

Resolve into partial fractions $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Solution:

This is an improper fraction first we convert it into a polynomial and a proper fraction by division.

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Let $\frac{8x - 4}{x^2 - 2x - 1} = \frac{8x - 4}{(3x + 1)(x - 1)} = \frac{A}{x - 1} + \frac{B}{3x + 1}$

Multiplying both sides by $(x - 1)(3x + 1)$ we get

$$8x - 4 = A(3x + 1) + B(x - 1) \quad (I)$$

Put $x - 1 = 0$, $\Rightarrow x = 1$ in (I), we get

The value of A

$$8(1) - 4 = A(3(1) + 1) + B(1 - 1)$$

$$8 - 4 = A(3 + 1) + 0$$

$$4 = 4A$$

\Rightarrow

$$A = 1$$

Put $3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$ in (I)

$$8\left(-\frac{1}{3}\right) - 4 = B\left(-\frac{1}{3} - 1\right)$$

$$-\frac{8}{3} - 4 = \left(-\frac{4}{3}\right)$$

$$-\frac{20}{3} = -\frac{4}{3} B$$

$$\Rightarrow B = \frac{20}{3} \times \frac{3}{4} = 5$$

Hence the required partial fractions are

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{1}{x-1} + \frac{5}{3x+1}$$

Example 5:

Resolve into partial fraction $\frac{8x - 8}{x^3 - 2x^2 - 8x}$

Solution: $\frac{8x - 8}{x^3 - 2x^2 - 8x} = \frac{8x - 8}{x(x^2 - 2x - 8)} = \frac{8x - 8}{x(x-4)(x+2)}$

Let $\frac{8x - 8}{x^3 - 2x^2 - 8x} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+2}$

Multiplying both sides by L.C.M. i.e., $x(x-4)(x+2)$

$$8x - 8 = A(x-4)(x+2) + Bx(x+2) + Cx(x-4)$$

(I)

Put $x = 0$ in equation (I), we have

$$8(0) - 8 = A(0-4)(0+2) + B(0)(0+2) + C(0)(0-4)$$

$$-8 = -8A + 0 + 0$$

$$\Rightarrow A = 1$$

Put $x - 4 = 0 \Rightarrow x = 4$ in Equation (I), we have

$$8(4) - 8 = B(4)(4+2)$$

$$32 - 8 = 24B$$

$$24 = 24B$$

$$\Rightarrow B = 1$$

Put $x + 2 = 0 \Rightarrow x = -2$ in Eq. (I), we have

$$8(-2) - 8 = C(-2)(-2-4)$$

$$-16 - 8 = C(-2)(-6)$$

$$-24 = 12C$$

$$\Rightarrow C = -2$$

Hence the required partial fractions

$$\frac{8x - 8}{x^3 - 2x^2 - 8x} = \frac{1}{x} - \frac{1}{x-4} - \frac{2}{x+2}$$

Exercise 4.1

Resolve into partial fraction:

Q.1 $\frac{2x + 3}{(x-2)(x+5)}$

Q.2 $\frac{2x + 5}{x^2 + 5x + 6}$

Q.3 $\frac{3x^2 - 2x - 5}{(x-2)(x+2)(x+3)}$

Q.4 $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Q.5 $\frac{x}{(x-a)(x-b)(x-c)}$

Q.6 $\frac{1}{(1-ax)(1-bx)(1-cx)}$

Q.7 $\frac{2x^3 - x^2 + 1}{(x+3)(x-1)(x+5)}$

Q.8 $\frac{1}{(1-x)(1-2x)(1-3x)}$

Q.9 $\frac{6x + 27}{4x^3 - 9x}$

Q.10 $\frac{9x^2 - 9x + 6}{(x-1)(2x-1)(x+2)}$

Q.11 $\frac{x^4}{(x-1)(x-2)(x-3)}$

Q.12 $\frac{2x^3 + x^2 - x - 3}{x(x-1)(2x+3)}$

Answers 4.1

Q.1 $\frac{1}{x-2} + \frac{1}{x+5}$

Q.2 $\frac{1}{x+2} + \frac{1}{x+3}$

Q.3 $\frac{3}{20(x-2)} - \frac{11}{4(x-2)} + \frac{28}{5(x+3)}$

Q.4 $1 + \frac{3}{x-4} - \frac{24}{x-5} + \frac{30}{x-6}$

Q.5 $\frac{a}{(a-b)(a-c)(x-a)} + \frac{b}{(b-a)(b-c)(x-b)} + \frac{c}{(c-b)(c-a)(x-c)}$

Q.6 $\frac{a^2}{(a-b)(a-c)(1-ax)} + \frac{b^2}{(b-a)(b-c)(1-bx)} + \frac{c^2}{(c-b)(c-a)(1-cx)}$

$$\text{Q.7} \quad 2 + \frac{31}{4(x+3)} + \frac{1}{12(x-1)} - \frac{137}{6(x+5)}$$

$$\text{Q.8} \quad \frac{1}{2(1-x)} - \frac{4}{(1-2x)} + \frac{9}{2(1-3x)}$$

$$\text{Q.9} \quad \frac{3}{x} + \frac{4}{2x-3} + \frac{2}{2x+3}$$

$$\text{Q.10} \quad \frac{2}{x-1} - \frac{3}{2x-1} + \frac{4}{x+12}$$

$$\text{Q.11} \quad x + 6 + \frac{1}{2(x-1)} - \frac{16}{x-2} + \frac{81}{2(x-3)}$$

$$\text{Q.12} \quad 1 + \frac{1}{x} - \frac{1}{5(x-1)} - \frac{8}{5(2x+3)}$$

4.9 Type II:

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\frac{x^2 - 3x + 1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

Multiplying both sides by L.C.M. i.e., $(x-1)^2(x-2)$, we get
 $x^2 - 3x + 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$ (I)

Putting $x-1=0 \Rightarrow x=1$ in (I), then

$$(1)^2 - 3(1) + 1 = B(1-2)$$

$$1 - 3 + 1 = -B$$

$$-1 = -B$$

$$\Rightarrow B = 1$$

Putting $x-2=0 \Rightarrow x=2$ in (I), then

$$(2)^2 - 3(2) + 1 = C(2-1)^2$$

$$4 - 6 + 1 = C(1)^2$$

$$\Rightarrow -1 = C$$

Now $x^2 - 3x + 1 = A(x^2 - 3x + 2) + B(x-2) + C(x^2 - 2x + 1)$

Comparing the co-efficient of like powers of x on both sides, we get

$$A + C = 1$$

$$A = 1 - C$$

$$= 1 - (-1)$$

$$= 1 + 1 = 2$$

$$\Rightarrow A = 2$$

Hence the required partial fractions are

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x-2}$$

Example 2:

Resolve into partial fraction $\frac{1}{x^4(x+1)}$

Solution

$$\frac{1}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$$

Where A, B, C, D and E are constants. To find these constants multiplying both sides by L.C.M. i.e., $x^4(x+1)$, we get

$$1 = A(x^3)(x+1) + Bx^2(x+1) + Cx(x+1) + D(x+1) + Ex^4$$

(I)

Putting $x = -1$ in Eq. (I)

$$1 = E(-1)^4$$

$$\Rightarrow E = 1$$

Putting $x = 0$ in Eq. (I), we have

$$1 = D(0+1)$$

$$1 = D$$

$$\Rightarrow D = 1$$

$$1 = A(x^4 + x^3) + B(x^3 + x^2) + C(x^2 + x) + D(x+1) + Ex$$

Comparing the co-efficient of like powers of x on both sides.

$$\text{Co-efficient of } x^3 : A + B = 0 \quad \dots\dots\dots$$

(i)

$$\text{Co-efficient of } x^2 : B + C = 0 \quad \dots\dots\dots$$

(ii)

$$\text{Co-efficient of } x : C + D = 0 \quad \dots\dots\dots$$

(iii)

Putting the value of $D = 1$ in (iii)

$$C + 1 = 0$$

$$\Rightarrow C = -1$$

Putting this value in (ii), we get

$$B - 1 = 0$$

$$\Rightarrow B = 1$$

Putting $B = 1$ in (i), we have

$$A + 1 = 0$$

$$\Rightarrow A = -1$$

Hence the required partial fraction are

$$\frac{1}{x^4(x+1)} = \frac{-1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x+1}$$

Example 3:

Resolve into partial fractions $\frac{4+7x}{(2+3x)(1+x)^2}$

Solution:

$$\frac{4+7x}{(2+3x)(1+x)^2} = \frac{A}{2+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

Multiplying both sides by L.C.M. i.e., $(2+3x)(1+x)^2$

We get $4+7x = A(1+x)^2 + B(2+3x)(1+x) + C(2+3x) \dots (I)$

$$\text{Put } 2+3x=0 \quad \Rightarrow \quad x = -\frac{2}{3} \text{ in (I)}$$

$$\text{Then } 4+7\left(-\frac{2}{3}\right) = A\left(1-\frac{2}{3}\right)^2$$

$$4 - \frac{14}{3} = A\left(-\frac{1}{3}\right)^2$$

$$-\frac{2}{3} = \frac{1}{9}A$$

$$\Rightarrow A = \frac{-2}{3} \times \frac{9}{1} = -6$$

$$A = -6$$

Put $1+x=0 \quad \Rightarrow \quad x=-1$ in eq. (I), we get

$$4+7(-1) = C(2-3)$$

$$4-7 = C(-1)$$

$$-3 = -C$$

$$\Rightarrow C = 3$$

$$4+7x = A(x^2+2x+1) + B(2+5x+3x^2) + C(2+3x)$$

Comparing the co-efficient of x^2 on both sides

$$A+3B=0$$

$$-6+3B=0$$

$$3B=6$$

$$\Rightarrow B=2$$

Hence the required partial fraction will be

$$\frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}$$

Exercise 4.2**Resolve into partial fraction:**

Q.1
$$\frac{x+4}{(x-2)^2(x+1)}$$

Q.2.
$$\frac{1}{(x+1)(x^2-1)}$$

Q.3
$$\frac{4x^3}{(x+1)^2(x^2-1)}$$

Q.4
$$\frac{2x+1}{(x+3)(x-1)(x+2)^2}$$

Q.5
$$\frac{6x^2-11x-32}{(x+6)(x+1)^2}$$

Q.6
$$\frac{x^2-x-3}{(x-1)^3}$$

Q.7
$$\frac{5x^2+36x-27}{x^4-6x^3+9x^2}$$

Q.8
$$\frac{4x^2-13x}{(x+3)(x-2)^2}$$

Q.9
$$\frac{x^4+1}{x^2(x-1)}$$

Q.10
$$\frac{x^3-8x^2+17x+1}{(x-3)^3}$$

Q.11
$$\frac{x^2}{(x-1)^3(x+2)}$$

Q.12
$$\frac{2x+1}{(x+2)(x-3)^2}$$

Answers 4.2

Q.1
$$-\frac{1}{3(x-2)} + \frac{2}{(x-2)^2} + \frac{1}{3(x+1)}$$

Q.2
$$\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}$$

Q.3
$$\frac{1}{2(x-1)} + \frac{7}{2(x+1)} - \frac{5}{(x+1)^2} + \frac{2}{(x+1)^3}$$

Q.4
$$\frac{5}{4(x+3)} + \frac{1}{12(x-1)} - \frac{4}{3(x+2)} + \frac{1}{(x+2)^2}$$

Q.5
$$\frac{10}{x+6} - \frac{4}{x+1} - \frac{3}{(x-1)^2}$$

Q.6
$$\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{3}{(x-1)^3}$$

Q.7
$$\frac{2}{x} - \frac{3}{x^2} - \frac{2}{(x-3)} + \frac{14}{(x-3)^2}$$

Q.8
$$\frac{3}{x+3} + \frac{1}{x-2} - \frac{2}{(x-2)^2}$$

$$\text{Q.9} \quad x + 1 - \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

$$\text{Q.10} \quad 1 + \frac{1}{x-3} - \frac{4}{(x-3)^2} + \frac{7}{(x-3)^3}$$

$$\text{Q.11} \quad \frac{4}{27(x-1)} + \frac{5}{9(x-1)^2} + \frac{1}{3(x-1)^3} - \frac{4}{27(x+2)}$$

$$\text{Q.12} \quad -\frac{3}{25(x+2)} + \frac{3}{25(x-3)} + \frac{7}{5(x-3)^2}$$

4.10 Type III:

When the denominator contains ir-reducible quadratic factors which are non-repeated.

Example 1:

Resolve into partial fractions $\frac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by L.C.M. i.e., $(x+3)(x^2+1)$, we get

$$9x-7 = A(x^2+1) + (Bx+C)(x+3) \quad (\text{I})$$

Put $x+3=0 \Rightarrow x=-3$ in Eq. (I), we have

$$9(-3)-7 = A((-3)^2+1) + (B(-3)+C)(-3+3)$$

$$-27-7 = 10A+0$$

$$A = -\frac{34}{10} \Rightarrow \boxed{A = -\frac{17}{5}}$$

$$9x-7 = A(x^2+1) + B(x^2+3x) + C(x+3)$$

Comparing the co-efficient of like powers of x on both sides

$$A+B=0$$

$$3B+C=9$$

Putting value of A in Eq. (i)

$$-\frac{17}{5} + B = 0 \Rightarrow \boxed{B = \frac{17}{5}}$$

From Eq. (iii)

$$C = 9 - 3B = 9 - 3\left(\frac{17}{5}\right)$$

$$= 9 - \frac{51}{5} \Rightarrow \boxed{C = -\frac{6}{5}}$$

Hence the required partial fraction are

$$\frac{-17}{5(x+3)} + \frac{17x-6}{5(x^2+1)}$$

Example 2:

Resolve into partial fraction $\frac{x^2+1}{x^4+x^2+1}$

Solution:

$$\text{Let } \frac{x^2+1}{x^4+x^2+1} = \frac{x^2+1}{(x^2-x+1)(x^2+x+1)}$$

$$\frac{x^2+1}{(x^2-x+1)(x^2+x+1)} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+x+1}$$

Multiplying both sides by L.C.M. i.e., $(x^2-x+1)(x^2+x+1)$
 $x^2+1 = (Ax+B)(x^2+x+1) + (Cx+D)(x^2-x+1)$

Comparing the co-efficient of like powers of x, we have

Co-efficient of x^3	:	$A + C = 0$	(i)
Co-efficient of x^2	:	$A + B - C + D = 1$	(ii)
Co-efficient of x	:	$A + B + C - D = 0$	(iii)
Constant		$B + D = 1$	(iv)

Subtract (iv) from (ii) we have

$$A - C = 0 \quad \text{.....} \quad \text{(v)}$$

$$A = C \quad \text{.....} \quad \text{(vi)}$$

Adding (i) and (v), we have

$$A = 0$$

Putting $A = 0$ in (vi), we have

$$C = 0$$

Putting the value of A and C in (iii), we have

$$B - D = 0 \quad \text{.....} \quad \text{(vii)}$$

Adding (iv) and (vii)

$$2B = 1 \quad \Rightarrow \quad B = \frac{1}{2}$$

from (vii) $B = D$, therefore

$$D = \frac{1}{2}$$

Hence the required partial fraction are

$$\frac{0x + \frac{1}{2}}{(x^2 - x + 1)} + \frac{0x + \frac{1}{2}}{(x^2 + x + 1)}$$

i.e., $\frac{1}{2(x^2 - x + 1)} + \frac{1}{2(x^2 + x + 1)}$

Exercise 4.3

Resolve into partial fraction:

Q.1 $\frac{x^2 + 3x - 1}{(x - 2)(x^2 + 5)}$

Q.2 $\frac{x^2 - x + 2}{(x + 1)(x^2 + 3)}$

Q.3 $\frac{3x + 7}{(x + 3)(x^2 + 1)}$

Q.4 $\frac{1}{(x^3 + 1)}$

Q.5 $\frac{1}{(x + 1)(x^2 + 1)}$

Q.6 $\frac{3x + 7}{(x^2 + x + 1)(x^2 - 4)}$

Q.7 $\frac{3x^2 - x + 1}{(x + 1)(x^2 - x + 3)}$

Q.8 $\frac{x + a}{x^2(x - a)(x^2 + a^2)}$

Q.9 $\frac{x^5}{x^4 - 1}$

Q.10 $\frac{x^2 + x + 1}{(x^2 - x - 2)(x^2 - 2)}$

Q.11 $\frac{1}{x^3 - 1}$

Q.12 $\frac{x^2 + 3x + 3}{(x^2 - 1)(x^2 + 4)}$

Answers 4.3

Q.1 $\frac{1}{x - 2} + \frac{3}{x^2 + 5}$

Q.2 $\frac{1}{x + 1} - \frac{1}{x^2 + 3}$

Q.3 $-\frac{1}{5(x + 3)} + \frac{x + 12}{5(x^2 + 1)}$

Q.4 $\frac{1}{3(x + 1)} - \frac{(x - 2)}{3(x^2 - x + 1)}$

Q.5 $\frac{1}{2(x + 1)} - \frac{x - 1}{2(x^2 + 1)}$

Q.6 $\frac{13}{28(X - 2)} - \frac{1}{12(X + 2)} - \frac{8X + 31}{21(X^2 + X + 1)}$

Q.7 $\frac{1}{x + 1} + \frac{2x - 2}{x^2 - x + 3}$

Q.8 $\frac{1}{a^3} \left[\frac{1}{X - a} + \frac{x}{X^2 + a^2} - \frac{2}{X} - \frac{a}{X^2} \right]$

$$\text{Q.9} \quad x + \frac{1}{4(x-1)} + \frac{1}{4(x+1)} - \frac{x}{2(x^2+1)}$$

$$\text{Q.10} \quad \frac{1}{3(x+1)} + \frac{7}{6(x-2)} - \frac{3x+2}{2(x^2-2)}$$

$$\text{Q.11} \quad \frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}$$

$$\text{Q.12} \quad \frac{7}{10(x-1)} - \frac{1}{10(x+1)} - \frac{3x-1}{5(x^2+4)}$$

4.11 Type IV: Quadratic repeated factors

When the denominator has repeated Quadratic factors.

Example 1:

Resolve into partial fraction $\frac{x^2}{(1-x)(1+x^2)^2}$

Solution:

$$\frac{x^2}{(1-x)(1+x^2)^2} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2}$$

Multiplying both sides by L.C.M. i.e., $(1-x)(1+x^2)^2$ on both sides, we have

$$x^2 = A(1+x^2)^2 + (Bx+C)(1-x)(1+x^2) + (Dx+E)(1-x) \quad \dots\dots(i)$$

$$x^2 = A(1+2x^2+x^4) + (Bx+C)(1-x+x^2-x^3) + (Dx+E)(1-x)$$

Put $1-x=0 \Rightarrow x=1$ in eq. (i), we have

$$(1)^2 = A(1+(1)^2)^2$$

$$1 = 4A \quad \Rightarrow \quad \boxed{A = \frac{1}{4}}$$

$$x^2 = A(1+2x^2+x^4) + B(x-x^2+x^3-x^4) + C(1-x+x^2-x^3) + D(x-x^2) + E(1-x) \quad \dots\dots(ii)$$

Comparing the co-efficients of like powers of x on both sides in

Equation (II), we have

$$\text{Co-efficient of } x^4 \quad : \quad A - B = 0 \quad \dots\dots(i)$$

$$\text{Co-efficient of } x^3 \quad : \quad B - C = 0 \quad \dots\dots(ii)$$

$$\text{Co-efficient of } x^2 \quad : \quad 2A - B + C - D = 1 \quad \dots\dots(iii)$$

$$\text{Co-efficient of } x \quad : \quad B - C + D - E = 0 \quad \dots\dots(iv)$$

$$\text{Co-efficient term} \quad : \quad A + C + E = 0 \quad \dots\dots(v)$$

$$\text{from (i),} \quad B = A$$

$$\Rightarrow \quad B = \frac{1}{4} \quad \because \quad A = \frac{1}{4}$$

from (i)

$$B = C$$

$$\Rightarrow C = \frac{1}{4} \quad \because C = \frac{1}{4}$$

from (iii)

$$\begin{aligned} D &= 2A - B + C - 1 \\ &= 2\left(\frac{1}{4}\right) - \frac{1}{4} + \frac{1}{4} - 1 \end{aligned}$$

$$\Rightarrow \boxed{D = -\frac{1}{2}}$$

from (v)

$$\begin{aligned} E &= -A - C \\ E &= -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \end{aligned}$$

Hence the required partial fractions are by putting the values of A, B, C, D, E,

$$\begin{aligned} &\frac{\frac{1}{4}}{1-x} + \frac{\frac{1}{4}x + \frac{1}{4}}{1+x^2} + \frac{-\frac{1}{2}x - \frac{1}{2}}{(1+x^2)^2} \\ &\frac{1}{4(1-x)} + \frac{(x+1)}{4(1+x^2)} - \frac{x+1}{2(1+x^2)^2} \end{aligned}$$

Example 2:

Resolve into partial fractions $\frac{x^2 + x + 2}{x^2(x^2 + 3)^2}$

Solution:

$$\text{Let } \frac{x^2 + x + 2}{x^2(x^2 + 3)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 3} + \frac{Ex + F}{(x^2 + 3)^2}$$

Multiplying both sides by L.C.M. i.e., $x^2(x^2 + 3)^2$, we have

$$\begin{aligned} x^2 + x + 2 &= Ax(x^2 + 3)^2 + B(x^2 + 3)^2 \\ &\quad + (Cx + D)x^2(x^2 + 3) + (Ex + F)(x^2) \end{aligned}$$

Putting $x = 0$ on both sides, we have

$$\begin{aligned} 2 &= B(0 + 3)^2 \\ 2 &= 9B \quad \Rightarrow \quad \boxed{B = \frac{2}{9}} \end{aligned}$$

$$\begin{aligned} \text{Now } x^2 + x + 2 &= Ax(x^4 + 6x^2 + 9) + B(x^4 + 6x^2 + 9) \\ &\quad + C(x^5 + 3x^2) + D(x^4 + 3x^2) + E(x^3) + Fx^2 \\ x^2 + x + 2 &= (A + C)x^5 + (B + D)x^4 + (6A + 3C + E)x^3 \end{aligned}$$

$$+(6B+3D+F)x^2 + (x+9B)$$

Comparing the co-efficient of like powers of x on both sides of Eq.

(I), we have

$$\text{Co-efficient of } x^5 \quad : \quad A + C = 0 \quad \dots\dots\dots$$

(i)

$$\text{Co-efficient of } x^4 \quad : \quad B - D = 0 \quad \dots\dots\dots$$

(ii)

$$\text{Co-efficient of } x^3 \quad : \quad 6A + 3C + E = 0 \quad \dots\dots\dots$$

(iii)

$$\text{Co-efficient of } x^2 \quad : \quad 6B + 3D + F = 1 \quad \dots\dots\dots$$

(iv)

$$\text{Co-efficient of } x \quad : \quad 9A = 1 \quad \dots\dots\dots$$

(v)

$$\text{Co-efficient term} \quad : \quad 9B = 1 \quad \dots\dots\dots$$

(vi)

$$\text{from (v)} \quad 9A = 1$$

$$\Rightarrow \quad \boxed{A = \frac{1}{9}}$$

$$\text{from (i)} \quad \begin{aligned} A + C &= 0 \\ C &= -A \end{aligned}$$

$$\Rightarrow \quad \boxed{C = -\frac{1}{9}}$$

$$\text{from (ii)} \quad \begin{aligned} B + D &= 0 \\ D &= -B \end{aligned}$$

$$\Rightarrow \quad \boxed{D = -\frac{2}{9}}$$

$$\text{from (iii)} \quad 6A + 3C + E =$$

$$6\left(\frac{1}{9}\right) + 3\left(-\frac{1}{9}\right) + E = 0$$

$$E = \frac{3}{9} - \frac{6}{9}$$

$$\Rightarrow \quad \boxed{E = -\frac{1}{3}}$$

$$\text{from (iv)} \quad 6B + 3D + F = 1$$

$$F = 1 - 6B - 3D$$

$$= 1 - 6\left(\frac{2}{9}\right) - 3\left(\frac{2}{9}\right)$$

$$= 1 - \frac{12}{9} + \frac{6}{9}$$

$$\Rightarrow \boxed{F = \frac{1}{3}}$$

Hence the required partial fractions are

$$\begin{aligned} & \frac{1}{9} + \frac{2}{9} + \frac{-\frac{1}{9}x - \frac{2}{9}}{x^2 + 3} + \frac{-\frac{1}{3}x + \frac{1}{3}}{(x^2 + 3)^2} \\ &= \frac{1}{9x} + \frac{2}{9x^2} - \frac{x+2}{9(x^2+3)} - \frac{x-1}{3(x^2+3)^2} \end{aligned}$$

Exercise 4.4

Resolve into Partial Fraction:

$$\text{Q.1 } \frac{7}{(x+1)(x^2+2)^2}$$

$$\text{Q.2 } \frac{x^2}{(1+x)(1+x^2)^2}$$

$$\text{Q.3 } \frac{5x^2+3x+9}{x(x^2+3)^2}$$

$$\text{Q.4 } \frac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$$

$$\text{Q.5 } \frac{2x^4-3x^2-4x}{(x+1)(x^2+2)^2}$$

$$\text{Q.6 } \frac{x^3-15x^2-8x-7}{(2x-5)(1+x^2)^2}$$

$$\text{Q.7 } \frac{49}{(x-2)(x^2+3)^2}$$

$$\text{Q.8 } \frac{8x^2}{(1-x^2)(1+x^2)^2}$$

$$\text{Q.9 } \frac{x^4+x^3+2x^2-7}{(x+2)(x^2+x+1)^2}$$

$$\text{Q.10 } \frac{x^2+2}{(x^2+1)(x^2+4)^2}$$

$$\text{Q.11 } \frac{1}{x^4+x^2+1}$$

Answers 4.4

$$\text{Q.1 } \frac{7}{9(x+1)} - \frac{7x-7}{9(x^2+2)} - \frac{7x-7}{3(x^2+2)^2}$$

$$\text{Q.2 } \frac{1}{4(1+x)} - \frac{x-1}{4(1+x^2)} + \frac{x-1}{2(1+x^2)^2}$$

$$\text{Q.3 } \frac{1}{x} - \frac{x}{x^2+3} + \frac{2x+3}{(x^2+3)^2}$$

$$\text{Q.4 } \frac{2}{x-1} + \frac{2x-1}{x^2+x+1} + \frac{3}{(x^2+x+1)^2}$$

$$\text{Q.5} \quad \frac{1}{3(x+1)} + \frac{5(x-1)}{3(x^2+2)} - \frac{2(3x-1)}{(x^2+1)^2}$$

$$\text{Q.6} \quad -\frac{2}{2x-5} + \frac{x+3}{1+x^2} + \frac{x-2}{(1+x^2)^2}$$

$$\text{Q.7} \quad \frac{1}{x-2} - \frac{x+2}{x^2+3} - \frac{7x+14}{(x^2+3)^2}$$

$$\text{Q.8} \quad \frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{1+x^2} - \frac{4}{(1+x^2)^2}$$

$$\text{Q.9} \quad \frac{1}{x+2} + \frac{2x-3}{(x^2+x+1)^2} - \frac{1}{x^2+x+1}$$

$$\text{Q.10} \quad \frac{1}{9(x^2+1)} - \frac{1}{9(x^2+4)} + \frac{2}{3(x^2+4)^2}$$

$$\text{Q.11} \quad -\frac{(x-1)}{2(x^2-x+1)} + \frac{(x+1)}{2(x^2+x+1)}$$

Summary

Let $N(x) \neq 0$ and $D(x) \neq 0$ be two polynomials. The $\frac{N(x)}{D(x)}$ is called a proper fraction if the degree of $N(x)$ is smaller than the degree of $D(x)$.

For example: $\frac{x-1}{x^2+5x+6}$ is a proper fraction.

Also $\frac{N(x^1)}{D(x)}$ is called an improper fraction of the degree of $N(x)$ is greater than or equal to the degree of $D(x)$.

For example: $\frac{x^5}{x^4-1}$ is an improper fraction.

In such problems we divide $N(x)$ by $D(x)$ obtaining a quotient $Q(x)$ and a remainder $R(x)$ whose degree is smaller than that of $D(x)$.

Thus $\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ where $\frac{R(x)}{D(x)}$ is proper fraction.

Types of proper fraction into partial fractions.

Type 1:

Linear and distinct factors in the $D(x)$

$$\frac{x - a}{(x + a)(x + b)} = \frac{A}{x + a} + \frac{B}{x + b}$$

Type 2:

Linear repeated factors in D(x)

$$\frac{x - a}{(x + a)(x^2 + b^2)} = \frac{A}{x + a} + \frac{Bx + C}{x^2 + b^2}$$

Type 3:

Quadratic Factors in the D(x)

$$\frac{x - a}{(x + a)(x^2 + b)^2} = \frac{A}{x + a} + \frac{Bx + C}{x^2 + b^2}$$

Type 4:

Quadratic repeated factors in D(x):

$$\frac{x - a}{(x^2 + a^2)(x^2 + b^2)} = \frac{Ax + B}{x^2 + a^2} + \frac{Cx + D}{x^2 + b^2} + \frac{Ex + F}{(x^2 + b^2)^2}$$

Short Questions:

Write the short answers of the following:

Q.1: What is partial fractions?

Q.2: Define proper fraction and give example.

Q.3: Define improper fraction and give one example:

Q.4: Resolve into partial fractions $\frac{2x}{(x-2)(x+5)}$

Q.5: Resolve into partial fractions: $\frac{1}{x^2-x}$

Q.6: Resolve $\frac{7x+25}{(x+3)(x+4)}$ into partial fraction.

Q.7: Resolve $\frac{1}{x^2-1}$ into partial fraction:

Q.8: Resolve $\frac{x^2+1}{(x+1)(x-1)}$ into partial fractions.

Q.9: Write an identity equation of $\frac{8x^2}{(1-x^2)(1+x^2)^2}$

Q.10: Write an identity equation of $\frac{2x+5}{x^2+5x+6}$

Q.11: Write identity equation of $\frac{x-5}{(x+1)(x^2+3)}$

Q.12: Write an identity equation of $\frac{6x^3+5x^2-7}{3x^2-2x-1}$

Q.13: Write an identity equation of $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Q.14: Write an identity equation of $\frac{x^5}{x^4-1}$

Q.15: Write an identity equation of $\frac{2x^4-3x^2-4x}{(x+1)(x^2+2)^2}$

Q16. Form of partial fraction of $\frac{1}{(x+1)(x-2)}$ is _____.

Q17. Form of partial fraction of $\frac{1}{(x+1)^2(x-2)}$ is _____.

Q18. Form of partial fraction of $\frac{1}{(x^2+1)(x-2)}$ is _____.

Q19. Form of partial fraction of $\frac{1}{(x^2+1)(x-4)^2}$ is _____.

Q20. Form of partial fraction of $\frac{1}{(x^3-1)(x^2+1)}$ is _____.

Answers

Q4. $\frac{4}{7(x-2)} - \frac{10}{7(x+5)}$

Q5. $\frac{-1}{x} + \frac{1}{x-1}$

Q6. $\frac{4}{x+3} + \frac{3}{x+4}$

Q7. $\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$

Q8. $1 + \frac{1}{x+1} + \frac{1}{x-1}$

Q9. $\frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2} + \frac{Ex+F}{(1+x^2)^2}$

Q10. $\frac{A}{x+2} + \frac{B}{x+3}$

Q11. $\frac{A}{x+1} + \frac{Bx+C}{x^2+3}$

Q12. $(2x+3) + \frac{A}{x-1} + \frac{B}{3x+1}$

Q13. $1 + \frac{A}{4-4} + \frac{B}{x-5} + \frac{C}{x-6}$

Q14. $x + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

Q15. $\frac{A}{x+1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$

Q16. $\frac{A}{x+1} + \frac{B}{x-2}$

Q17. $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$

Q18. $\frac{Ax+B}{x^2+1} + \frac{C}{x-2}$

Q19. $\frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$

Q20. $\frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)} + \frac{Dx+E}{x^2+1}$

Objective Type Questions

- Q.1** Each questions has four possible answers. Choose the correct answer and encircle it.
- __1. If the degree of numerator $N(x)$ is equal or greater than the degree of denominator $D(x)$, then the fraction is:
 (a) proper (b) improper
 (c) Neither proper non-improper (d) Both proper and improper
- __2. If the degree of numerator is less than the degree of denominator, then the fraction is:
 (a) Proper (b) Improper
 (c) Neither proper non-improper (d) Both proper and improper
- __3. The fraction $\frac{2x + 5}{x^2 + 5x + 6}$ is known as:
 (a) Proper (b) Improper
 (c) Both proper and improper (d) None of these
- __4. The number of partial fractions of $\frac{6x + 27}{4x^3 - 9x}$ are:
 (a) 2 (b) 3
 (c) 4 (d) None of these
- __5. The number of partial fractions of $\frac{x^3 - 3x^2 + 1}{(x - 1)(x + 1)(x^2 - 1)}$ are:
 (a) 2 (b) 3
 (c) 4 (d) 5
- __6. The equivalent partial fraction of $\frac{x + 11}{(x + 1)(x - 3)^2}$ is:
 (a) $\frac{A}{x + 1} + \frac{B}{(x - 3)^2}$ (b) $\frac{A}{x + 1} + \frac{B}{x - 3}$
 (c) $\frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2}$ (d) $\frac{A}{x + 1} + \frac{Bx + C}{(x - 3)^2}$
- __7. The equivalent partial fraction of $\frac{x^4}{(x^2 + 1)(x^2 + 3)}$ is:
 (a) $\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 3}$ (b) $\frac{Ax + B}{x^2 + 1} + \frac{Cx}{x^2 + 3}$
 (c) $1 + \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 3}$ (d) $\frac{Ax}{x^2 + 1} + \frac{Bx}{x^2 + 3}$

__8. Partial fraction of $\frac{2}{x(x+1)}$ is:

(a) $\frac{2}{x} - \frac{1}{x+1}$

(b) $\frac{1}{x} - \frac{2}{x+1}$

(c) $\frac{2}{x} - \frac{2}{x+1}$

(d) $\frac{2}{x} + \frac{2}{x+1}$

__9. Partial fraction of $\frac{2x+3}{(x-2)(x+5)}$ is called:

(a) $\frac{2}{x-2} + \frac{1}{x+5}$

(b) $\frac{3}{x-2} + \frac{1}{x+5}$

(c) $\frac{2}{x-2} + \frac{3}{x+5}$

(d) $\frac{1}{x-2} + \frac{1}{x+5}$

__10. The fraction $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$ is called:

(a) Proper

(ii) Improper

(c) Both proper and Improper

(iv) None of these

Answers:

- | | | | | | | | | | |
|----|---|----|---|----|---|----|---|-----|---|
| 1. | b | 2. | a | 3. | a | 4. | b | 5. | c |
| 6. | c | 7. | c | 8. | c | 9. | d | 10. | B |