Chapter : Integration
Marks: 22

Integration of Standard Functions Formulae

1. $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. $\int a^x dx = \frac{a^x}{\log a}$
3. $\int e^x dx = e^x$
4. $\int \frac{1}{x} dx = \log x$
5. $\int \sin x dx = -\cos x$
6. $\int \cos x dx = \sin x$
7. $\int \tan x dx = \log(\sec x)$
8. $\int \cot x dx = \log(\sin x)$
9. $\int \sec x dx = \log(\sec x + \tan x)$
10. $\int \operatorname{cosec} x dx = \log(\operatorname{cosec} x - \cot x)$
11. $\int \sec^2 x dx = \tan x$
12. $\int \operatorname{cosec}^2 x dx = -\cot x$
13. $\int \sec x \tan x dx = \sec x$
14. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x$
15. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$
16. $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
17. $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\left(\frac{a + x}{a - x}\right)$
19. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left(\frac{x - a}{x + a}\right)$

20. $\int \frac{1}{\sqrt{a^2+x^2}}dx = \log\left(x+\sqrt{a^2+x^2}\right)$

21. $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left(x+\sqrt{x^2-a^2}\right)$

22. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

23. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)$

24. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)$

Composite Integrals

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1} \times \frac{1}{a}$
2. $\int \frac{dx}{(ax+b)^n} = \int (ax+b)^{-n} \times dx = \frac{(ax+b)^{-n+1}}{-n+1} \times \frac{1}{a}$
3. $\int \frac{dx}{ax+b} = \log(ax+b) \times \frac{1}{a}$
4. $\int e^{ax+b} dx = e^{ax+b} \times \frac{1}{a}$
5. $\int A^{ax+b} dx = \frac{A^{ax+b}}{\log_e A} \times \frac{1}{a}$
6. $\int \cos(ax+b)dx = \sin(ax+b) \times \frac{1}{a}$
7. $\int \sin(ax+b)dx = -\cos(ax+b) \times \frac{1}{a}$
8. $\int \sec^2(ax+b)dx = \tan(ax+b) \times \frac{1}{a}$
9. $\int \operatorname{cosec}^2(ax+b)dx = -\cot(ax+b) \times \frac{1}{a}$
10. $\int \sec(ax+b) + \tan(ax+b)dx = -\sec(ax+b) \times \frac{1}{a}$
11. $\int \operatorname{cosec}(ax+b) + \cot(ax+b)dx = -\operatorname{cosec}(ax+b) \times \frac{1}{a}$

12. $\int \frac{dx}{\sqrt{1-(ax+b)^2}} = \sin^{-1}(ax+b) \times \frac{1}{a}$

$\int \frac{dx}{\sqrt{1-(ax+b)^2}} = -\cos^{-1}(ax+b) \times \frac{1}{a}$

13. $\int \frac{dx}{\sqrt{1+(ax+b)^2}} = \tan^{-1}(ax+b) \times \frac{1}{a}$

$\int \frac{dx}{\sqrt{1+(ax+b)^2}} = -\cot^{-1}(ax+b) \times \frac{1}{a}$

Rules of Integration

1. $\int K dx = Kx$ Where K is constant
2. $\int Kf(x)dx = K\int f(x)dx$
3. $\int [f(x) \pm g(x)]dx = \int \left[f(x)dx \pm \int g(x)dx\right]$

Methods of Integration

Method 1: Simple or Direct Integration

Ex. 1: Evaluate $\int (5x^2+1)dx$

Soln. $\int (5x^2+1)dx = \int 5x^2 dx + \int 1 dx + c$

$= 5\int x^2 dx + x + c$

$= 5\left(\frac{x^{2+1}}{2+1}\right) + x + c$

$= \frac{5}{3}x^3 + x + x + c$ Where C is Constant of integration.

Ex. 2: Evaluate $\int (5^x + \sec^2 x)dx$

Soln. $\int (5^x + \sec^2 x)dx = \int 5^x dx + \int \sec^2 x dx$

$= \frac{5^x}{\log 5} + \tan x + c$

Ex. 3: Evaluate $\int (\tan x + 4\sin x)dx$

Soln. $\int (\tan x + 4\sin x)dx = \int \tan x dx + 4\int \sin x dx$

$$= \log(\sec x) + 4(-\cos x) + c$$
$$= \log(\sec x) - 4\cos x + c$$

Ex. 4: Evaluate $\int \sqrt[3]{x} \cdot x^2 dx$

Soln. $\int \sqrt[3]{x} \cdot x^2 dx$

Integration is not separable in multiplication $\therefore$ by law of indices.

$$= \int \left(x^{\frac{1}{3}} \cdot x^2 \right) dx$$

$$= \int x^{\frac{7}{3}} dx = \frac{x^{\left(\frac{7}{3}\right)+1}}{\left(\frac{7}{3}\right)+1} + c$$

$$\because \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= \frac{x^{\frac{10}{3}}}{\frac{10}{3}} + c \quad = \frac{3}{10} x^{\frac{10}{3}} + c$$

Ex. 5: Evaluate $\int \frac{1}{4 + x^2} dx$

Soln. $= \int \frac{1}{4 + x^2} dx$

$$= \int \frac{1}{(2)^2 + x^2} dx$$

$$\because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2} + c$$

Ex. 6: Evaluate $\int \frac{1}{x^2 - 7} dx$

Soln $= \int \frac{1}{x^2 - 7} dx$

$$= \int \frac{1}{x^2 - \left(\sqrt{7}\right)^2} dx$$

$$\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left(\frac{x - a}{x + a} \right)$$

=

Ex. 7: Evaluate $\int \frac{1}{25+x}dx$

Soln = $\int \frac{1}{25+x}dx$

By Formula $\int \frac{1}{a \pm bx}dx = \frac{1}{\pm b}\log(a \pm bx)$

$$\therefore \int \frac{1}{25+x} = \frac{1}{1}\log(25+x)+c$$

$$= \log(25+x)+c$$

HW

Ex. 8: Evaluate $\int \frac{1}{4+9x^2}dx$

Soln:

= $\int \frac{1}{4+9x^2}dx$

Taking 9 common from denominator to adjust std. formulae

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HWEx. 9: Evaluate $\int (x^2+1)^2 dx$

Soln: = $\int (x^2+1)^2 dx$

Expand $(a+b)^2 =$

=

=

Ex. 10: Evaluate $\int (3x+1)^5 dx$

Soln: = $\int (3x+1)^5 dx$

We have $\because \int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int f(ax)dx = \frac{g(ax)}{a}$

$$\therefore \int (3x+1)^5 dx = \left[\frac{(3x+1)^{5+1}}{(5+1)\times(3)}\right]$$

$$= \frac{(3x+1)^6}{6\times 3}$$

$$= \frac{1}{18}(3x+1)^6 + c$$

HW Ex. 11: Evaluate $\int \sqrt{2-5x}dx$

Soln:

$\int \sqrt{2-5x}dx$ =

=

=

=

HW Ex. 12: Evaluate $\int \cos(5x+2)dx$

Soln:

$\int \cos(5x+2)dx$ =

HW Ex. 13: Evaluate $\int \frac{1+x-x^2}{\sqrt{x}}dx$

Soln:

$$= \int \frac{1+x-x^2}{\sqrt{x}}dx$$

$$= \int \left\{\frac{1}{\sqrt{x}} + \frac{x}{x^{1/2}} - \frac{x^2}{x^{1/2}}\right\}$$

= … **By** $\frac{a^m}{a^n} = a^{m-n}$

= … $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$

$$= 2\sqrt{x} + \frac{2}{3}\cdot x^{3/2} - \frac{2}{3}\cdot x^{5/2} + c$$

Ex.14 : Evaluate $\int x(x-1)^2 dx$

Soln:

$=\int x(x-1)^2\,dx$

= ...Using $(a-b)^2 = a^2 - 2ab + b^2$

=

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=

Ex.15: Evaluate $\int \frac{x+1}{\sqrt{x-2}}\,dx$

Soln:

$=\int \frac{x+1}{\sqrt{x-2}}\,dx$

$=\int \frac{x-2+3}{\sqrt{x-2}}\,dx$ ...Note adjusting the constant in the numerator

= ...Resolving into fraction

=

=

=

=

$=\frac{2}{3}\sqrt{x-2}\,[x+7]+c$

HW Ex.16: Evaluate $\int \frac{dx}{\sqrt{5-4x^2}}$

Soln: To make coefficient of x^2 unity (1), taking 4 out of radical sign, we get

$I = \frac{1}{2}\int \frac{dx}{\sqrt{\frac{5}{4}-x^2}}$

=

=

=

HW Ex.17: Evaluate $\int \frac{1}{25-9x^2}dx$ **(CE) (S-05)**

Soln: Making coefficient of x^2 unity (1), we get

$\therefore$ I =

=

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HW Ex.18: Evaluate $\int \frac{dx}{\sqrt{4x^2+9}}dx$

Soln:

Ans is I = $\frac{1}{2}\cdot\log\left|2x+\sqrt{4x^2+9}\right|+c$ Where c = New constant = c_1-$\frac{1}{2}\log 2$

Ex.: Evaluate $\int\left(\frac{1}{\sqrt{4-9x^2}}+\frac{1}{3+2x^2}\right)dx$

Soln:

$$= \int \frac{dx}{\sqrt{4-9x^2}} + \int \frac{dx}{2x^2+3}$$ **…** Note the use of rule of integration

Making coefficient of x^2 unity (1) in both integrals, we get

I =

=

Note this step converting integral into their respective standard forms. Now, by the results of $\int \frac{dx}{\sqrt{a^2-x^2}}$ and $\int \frac{dx}{x^2+a^2}$, we get

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=

To Solve for Home Work.

Evaluate $\int \cos^3 x dx$ **(ME/CH) (S-05)**

Evaluate $\int \frac{dx}{3x-2}$ **(CE) (S-04)**

Evaluate $\int \frac{dx}{\sqrt{x+1}+\sqrt{x+2}}$ **(CM/IF) (W-05)**

Evaluate $\int x^2 \sqrt{x} dx$ **(ET) (S-04)**

Evaluate $\int \frac{1}{a^2+b^2x^2} dx$ **(CE) (S-06), (EE) (S-05)**

Method 2: Integration of Rational Functions

1. $\frac{x}{x+1}$ Can be expressed as $\frac{x+1-1}{x+1}$ which in turn equivalent to $1-\frac{1}{x+1}$

2. $\frac{3x+5}{2x-1}$ Can be expressed as $Q+\frac{R}{D}$ i.e. Quotient + $\frac{\text{Re}\,manider}{Divisor}$ as below, by actual division

$$2x-1\overline{)\,3x+5\,}\quad \text{quotient } 3/2$$

$$3x-\frac{3}{2}$$

$$-\;+$$

$$\frac{13}{2}$$

Thus $\dfrac{3x-5}{2x-1}=\dfrac{3}{2}+\dfrac{13/2}{2x-1}$

Integrate w.r.to x the following

A. $\dfrac{x+5}{x-2}$ **B.** $\dfrac{3x+5}{x+3}$ **C.** $\dfrac{x^2-1}{x^2+1}$ **D.** $\dfrac{5x+3}{3x-2}$ **E.** $\dfrac{2x^2+5x-1}{3x+2}$

A. $\dfrac{x+5}{x-2}$

Solution

Let I $= \displaystyle\int \frac{x+5}{x-2}dx$

Note that coefficient of x in numerator and denominator is same and therefore we need to manipulate only constant term of numerator as below.

$\therefore$ I $= \displaystyle\int \frac{x-2+7}{x-2}dx$

$= \displaystyle\int \left(\frac{x-2}{x-2}+\frac{7}{x-2}\right)dx$ Resolving into fractious

$= \displaystyle\int 1dx + 7\int \frac{dx}{x-2}$

$= x + 7\log|x-2| + C$

B. $\dfrac{3x+5}{x+3}$

Solution

Let I $= \displaystyle\int \frac{3x+5}{x+3}dx$

=

=

=

=

=

C. $\dfrac{x^2-1}{x^2+1}$

Solution

Let I $=\displaystyle\int \frac{x^2-1}{x^2+1}dx$

=

=

=

=

D. $\dfrac{5x+3}{3x-2}$

Solution

Let I $=\displaystyle\int \frac{5x+3}{3x-2}dx$

Note that coefficient of x in numerator and denominator are not equal. It is convenient to express it as sum of functions by actual division.

I $=\displaystyle\int\left(\frac{5}{3}+\frac{\frac{19}{3}}{3x-2}\right)dx$

$$3x-2\overline{)5x+3}\quad \frac{5}{3}$$
$$5x-\frac{10}{3}$$
$$-\;+$$
$$\frac{19}{3}\rightarrow \text{Remainder}$$

$=\dfrac{5}{3}\displaystyle\int dx+\frac{19}{3}\int\frac{dx}{3x-2}$

$=\dfrac{5}{3}x+\dfrac{19}{3}\cdot\log|3x-2|\cdot\dfrac{1}{3}+C$

$=\dfrac{5x}{3}+\dfrac{19}{9}\cdot\log|3x-2|+C$

E. $\dfrac{2x^2+5x-1}{3x+2}$

Solution

Let I $=\displaystyle\int \frac{2x^2+5x-1}{3x+2}dx$

Expressing the integrand as sum of functions by actual division as shown alongside, We get

$$==\int\left(\frac{2}{3}x+\frac{11}{9}+\frac{-\frac{13}{9}}{3x-2}\right)dx$$

$$\begin{array}{r} \frac{2}{3}x+\frac{11}{9} \\ 3x+2\,\big)\,2x^2+5x+1 \\ 2x^2+\frac{4}{3}x \\ -+ \\ \hline \frac{11x}{3}+1 \\ \frac{11x}{3}+\frac{22}{9} \\ -- \\ \hline -\frac{13}{9} \end{array}$$

$$=\frac{2}{3}\int xdx+\frac{11}{9}\int dx-\frac{13}{9}\int\frac{dx}{3x+2}$$

$$=\frac{2}{3}\cdot\frac{x^2}{2}+\frac{11}{9}x-\frac{13}{9}\log|3x+2|\cdot\frac{1}{3}+C$$

$$=\frac{x^2}{3}+\frac{11}{9}x-\frac{13}{27}\log|3x+2|+C$$

Method3: Integration by Trigonometric Transformation

In This case the integrand is expressed as sum of difference of functions by trigonometric transformation in which each function is in standard form those whose integral is known.

Trigonometric Formulae required for integration

$\sin^2 A+\cos^2 A=1$

$\cos^{2}— = 1-\sin^{2}—$

$\sin^{2}— = 1-\cos^{2}—$

$1+\tan^{2}— = Sec^{2}—$

$\tan^{2}— = Sec^{2}— -1$

$1+\cot^{2}— = \cos ec^{2}—$

$\cot^{2}— = \cos ec^{2}— -1$

$\sin(-—) = -\sin —$

$\cos(-—) = \cos —$

$\sin 2— = 2\sin — \cos —$

$\sin — = 2\sin\frac{—}{2}\cos\frac{—}{2}$

$Cos2— = Cos^{2}— - Sin^{2}—$

$Cos2— = 1-2Sin^{2}—$

$Cos2— = 2Cos^{2}— -1$

$$\sin^{2}— = \frac{1-\cos 2—}{2}$$

$$\cos^{2}— = \frac{1+\cos 2—}{2}$$

$$1-Cos2— = 2Sin^{2}—$$

$$1+Cos2— = 2\cos^{2}—$$

$$1-Cos— = 2Sin^{2}\frac{—}{2}$$

$$1+Cos— = 2\cos^{2}\frac{—}{2}$$

$$\sin 2— = \frac{2\tan —}{1+\tan^{2}—}$$

$$\cos 2— = \frac{2\tan —}{1-\tan^{2}—}$$

$$Sin3— = 3Sin— - 4Sin^{3}—$$

$$Cos3— = 4Cos^{3}— - 3Cos—$$

$$Sin(A+B) = SinA.CosB + CosA.SinB$$

$$Sin(A-B) = SinA.CosB - CosA.SinB$$

$$Cos(A+B) = CosA.CosB - SinA.SinB$$

$$Cos(A-B) = CosA.CosB + SinA.SinB$$

$$SinC + SinD = 2Sin(\frac{C+D}{2}).Cos(\frac{C-D}{2})$$

$$SinC - SinD = 2Cos(\frac{C+D}{2}).Sin(\frac{C-D}{2})$$

$$CosC + CosD = 2Cos(\frac{C+D}{2}).Cos(\frac{C-D}{2})$$

$$CosC - CosD = -2Sin(\frac{C+D}{2}).Sin(\frac{C-D}{2})$$

$$CosC - CosD = 2Sin(\frac{C+D}{2}).Sin(\frac{D-C}{2})$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$$

Integrate w.r.to x the following:

A. $\sin^2 x$ **B.** $\tan^2 x$ **C.** $\sin^2 x\cos^2 x$ **D.** $\dfrac{1}{\sin^2 x\cos^2 x}$

E. $\dfrac{1}{1-\sin x}$ **F.** $\sqrt{1+\sin 2x}$ **G.** $\sin 3x\cdot\cos 4x$

A. $\sin^2 x$

Solution

Let I $= \int \sin^2 dx$

Using $\sin^{2-} = \dfrac{1-\cos 2^{-}}{2}$

$\therefore$ I $= \int \dfrac{1-\cos 2x}{2}dx$

$= \dfrac{1}{2}\left[\int dx - \int \cos e 2x dx\right]$

$= \dfrac{1}{2}\left[x - \dfrac{\sin 2x}{2}\right] + C$

$= \dfrac{1}{2}x - \dfrac{1}{4}\sin 2x + C$

B. $\tan^2 x$

Solution

Let I $= \int \tan^2 x dx$

We have $1+\tan^{2-} = \sec^{2-}$ $\therefore$ $\tan^{2-} = \sec^{2-} - 1$

$\therefore$ I =

=

=

C. $\sin^2 x\cos^2 x$

Solution

Let I $= \int \sin^2 \cdot \cos^2 x dx$

$= \int (\sin x\cdot\cos x)^2 dx$ **...**Use of law of indices

$= \int \left\{\dfrac{2\sin x\cdot\cos x}{2}\right\}^2 dx$

$= \int \left(\dfrac{\sin 2x}{2}\right)^2 dx$

=

=

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=

D. $\frac{1}{\sin^2 x \cos^2 x}$

Solution

Let I $= \int \frac{1}{\sin^2 \cdot \cos^2 x} dx$

= **…**Note substitution for 1 in numerator

=

=

=

E. $\frac{1}{1-\sin x}$

Solution

Multiplying and dividing the integrand by $1+\sin x$ the conjugate of denominator, we get

Let I $= \int \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x} dx$

$= \int \frac{1+\sin x}{1-\sin^2 x} dx$

=

=

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=

F. $\sqrt{1+\sin 2x}$

Solution

Let I $= \int \sqrt{1+\sin 2x} dx$

Since $1+\sin 2x = \sin^2 x + \cos^2 x + 2\sin x \cdot \cos x$

$= (\sin x + \cos x)^2$

$\therefore$ I =

=

=

=

=

G. $\sin 3x \cdot \cos 4x$

Solution

Let I $= \int \sin 3x \cdot \cos 4x dx$

$= \frac{1}{2}\int 2 \cdot \sin 3x \cdot \cos 4x dx$

Using $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$, we get

$= \frac{1}{2}\int \{\sin(3x+4x)\sin(3x-4x)\}dx$

$= \frac{1}{2}\int (\sin 7x - \sin x)dx \qquad \ldots \sin(-x) = -\sin x$

$= \frac{1}{2}\int \sin 7x - \frac{1}{2}\int \sin x dx$

$= \frac{1}{2} \cdot \frac{-\cos 7x}{7} - \frac{1}{2} \cdot -\cos x + C$

$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + C$

Integrate w.r.to x the following

A. $\frac{1}{1-\cos x}$ **B.** $\frac{1-\cos x}{1+\cos x}$ **C.** $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$

D $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$

A. $\frac{1}{1-\cos x}$

Solution

Let I $= \int \frac{1}{1-\cos x} dx$

$$=\int \frac{1}{2\cdot \sin^2 \frac{x}{2}}dx$$

$$=\frac{1}{2}\int \cos ec^2 \frac{x}{2}dx$$

$$=\frac{1}{2}\cdot -\frac{\cot \frac{x}{2}}{\frac{1}{2}}+C$$

$$=-\cot\frac{x}{2}+C$$

B. $\dfrac{1-\cos x}{1+\cos x}$

Solution

Let I $=\int \frac{1-\cos x}{1+\cos x}$

=

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=

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=

C. $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

Solution

Let I =

=

=

= ... $\because \quad \tan^{-1}(\tan^{-})=^{-}$

=

=

D $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$

Solution

Let I =

=

=

=

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=

Methods of Integration

1. **Integration by Substitution.**
2. **Integration by Parts.**
3. **Integration by Partial Fractions.**

1. **Integration by Substitution.**

Ex. Evaluate $\int \sin^3 \cdot \cos x dx$

Solution:

Let I $= \int \sin^3 x \cdot \cos x dx$ **...** (1)

Putting $\sin x = t$ differentiating x w.r.to t, we get

$\therefore \quad \cos x dx = dt$

Then integral (1) becomes,

$= \int t^3 dt$

$= \dfrac{t^4}{4} + C = \dfrac{1}{4}\sin^4 x + C$

Ex. Evaluate $\int (x^2+1)^2 \cdot x \cdot dx$

Solution:

Let I $= \int (x^2+1)^2 \cdot x \cdot dx$ **...** (1)

Putting $x^2 + 1 = t$ differentiating x w.r.to t, we get

$\therefore \quad 2x dx = dt$ OR $x \cdot dx = \dfrac{dt}{2}$

Then integral (1) becomes,

Let I $= \int t^2 \cdot \dfrac{dt}{2} dt$

$$=\frac{1}{2}\int t^2 \cdot dt$$

$$=\frac{1}{2}\cdot\frac{t^3}{3}+C=\frac{1}{6}\cdot\left(x^2+1\right)^3+C$$

Ex. Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

Solution:

Let I $= \int \cos\sqrt{x}\cdot\frac{1}{\sqrt{x}}dx$

Putting $\sqrt{x}=t$ differentiating x w.r.to t, we get

$\therefore$ $\frac{1}{2\sqrt{x}}dx=dt$ OR $\frac{1}{\sqrt{x}}dx=2dt$

Then the integral reduce to

Let I $=\int \cos t\cdot 2dt$

$$=2\int \cos t\cdot dt$$

$$=2\sin t+C=2\sin\sqrt{x}+C$$

Ex. Evaluate $\int \frac{\left(\tan^{-1}x\right)^2}{1+x^2}dx$

Solution:

Let I $= \int \frac{\left(\tan^{-1}x\right)^2}{1+x^2}dx$

$$=\int \left(\tan^{-1}x\right)^2\cdot\frac{1}{1+x^2}dx \qquad \textbf{...} \ (1)$$

Putting $\tan^{-1}x=t$ differentiating x w.r.to t, we get

$\therefore$ $\frac{1}{1+x^2}dx=dt$

Then integral (1) becomes,

Let I =

=

=

Ex. Evaluate $\int \frac{\cos x}{(1+\sin x)^{3/2}}dx$

Solution:

Let I $= \int \frac{\cos x}{(1+\sin x)^{3/2}} dx$ ... (1)

Putting ------------------------------------ & differentiating x w.r.to t, we get

$\therefore$ ---------------------------- $= dt$

Then integral (1) becomes,

Let I =

=

=

=

Ex. Evaluate $\int \frac{\sec x \cdot \cos ecx}{\log(\tan x)} dx$

Solution:

Let I $= \int \frac{\sec x \cdot \cos ecx}{\log(\tan x)} dx$ ... (1)

(Since $\frac{d}{dx}[\log(\tan x)] = \frac{1}{\tan x} \cdot \sec^2 x$

$= \sec x \cdot \cos ecx =$ Numerator of integrand)

Let I =

=

=

HWEx. Evaluate $\int \frac{dx}{x \cdot \log x \cdot \log(\log x)}$

Solution:

Let I $= \int \frac{dx}{x \cdot \log x \cdot \log(\log x)}$ ... (1)

Ex. Evaluate $\int \frac{dx}{x+\sqrt{x}}$

Solution:

Let I $= \int \frac{dx}{x+\sqrt{x}}$

$= \int \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$ **...** Note this step $x = \sqrt{x} \cdot \sqrt{x}$

$= \int \frac{1}{\sqrt{x}+1} \cdot \frac{1}{\sqrt{x}} dx$ **...** (1)

Important Deductions

$$\int \frac{f'(x) \cdot dx}{f(x)} = \log\left|f(x)\right| + C$$

Ex. Evaluate $\int \frac{dx}{x \cdot \log x}$

Solution:

Let I =

=

=

=

Ex. Evaluate $\int \frac{1+\tan x}{1-\tan x}dx$

Solution:

Let I $= \int \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}dx$ **...** Note the substitution $\tan x = \frac{\sin x}{\cos x}$

I =

=

Ex. Evaluate $\int \frac{1}{1+\tan x}dx$

Solution:

Let I $= \int \frac{1}{1+\frac{\sin x}{\cos x}}dx$

$= \int \frac{\cos x}{\cos x + \sin x}dx$

$= \frac{1}{2}\int \frac{2\cos x}{\cos x + \sin x}dx$ **…**Note this step of multiplying and dividing by 2

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x + \sin x}dx$$

=

I =

=

=

Ex. Evaluate $\int \frac{x}{a^2 - x^2}dx$

Solution:

Let I $= \int \frac{x}{a^2 - x^2}dx$

Putting $f(x) = a^2 - x^2$ $\therefore$ $f'(x) = -2x$

Let I =

=

=

Integrand of the form		**Suggested substitution**
1.	$f(x^n)\cdot x^{n-1}$	$x^n = t$
2.	$[f(x)]^n \cdot f'(x)$	$f(x) = t$
3.	$\frac{f'(x)}{[f(x)]^n}$	$f(x) = t$
4.	$\frac{f'(x)}{f(x)}$	$f(x) = t$
5.	$\frac{f'(x)}{\sqrt{f(x)}}$	$f(x) = t$
6.	$f(\log x)\cdot \frac{1}{X}$	$\log x = t$
7.	$f(\sin x)\cdot \cos x$	$\sin x = t$

8.	$f(\cos x)\cdot\sin x$	$\cos x = t$
9.	$f(\tan x)\cdot\sec^2 x$	$\tan x = t$
10.	$\sqrt{a^2 - x^2}$	$x = a\sin^{-}$ or $x = a\cos^{-}$
11.	$\sqrt{x^2 - a^2}$	$x = a\sec^{-}$ or $x = a\cos ec^{-}$
12.	$\sqrt{x^2 + a^2}$ OR $x^2 + a^2$	$x = a\tan^{-}$ or $x = a\cot^{-}$
13.	$\sqrt{\dfrac{a-x}{a+x}}$	$x = a\cos^{-}$
14.	$\sqrt{\dfrac{a-x}{x}}$	$x = a\sin^{2-}$
15.	$\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos^2 2^{-}$
16.	$\sqrt{2ax - x^2}$	$x = 2a\sin^{2-}$

Integral of the form

$\int f(x)\cdot\sqrt{ax+b}\,dx$ And

$\int \dfrac{f(x)}{\sqrt{ax+b}}dx$ Where $f(x)$ is a polynomial in x

Ex. Evaluate $\int x^2\sqrt{x+1}\,dx$

Solution:

Let I $= \int x^2\sqrt{x+1}\,dx$ ... (1)

Putting $x+1=t$ $\therefore$ $x=t+1$ and differentiating x w.r.to t, we get $dx = dt$

Then integral (1) becomes

Let I $= \int (t-1)^2\sqrt{t}\,dt$

$= \int \left(t^2 - 2t + 1\right)\cdot t^{1/2}dt$

$= \int \left(t^2\cdot t^{1/2} - 2t\cdot t^{1/2} + t^{1/2}\right)dt$

$$=\int t^{5/2}dt-2\int t^{3/2}dt+\int t^{1/2}dt$$

$$=\frac{t^{7/2}}{\frac{7}{2}}-2\frac{t^{5/2}}{\frac{5}{2}}+1\frac{t^{3/2}}{\frac{3}{2}}+C$$

$$=\frac{2}{7}(x+1)^{7/2}-\frac{4}{5}(x+1)^{5/2}+\frac{2}{3}(x+1)^{3/2}+C$$

Ex. Evaluate $\int\frac{3x+4}{\sqrt{x+3}}dx$

Solution:

Let I $=\int\frac{3x+4}{\sqrt{x+3}}dx$ **...** (1)

Putting $x+3=t$ $\therefore$ $x=3+t$ and differentiating x w.r.to t, we get $dx=dt$

Then integral (1) becomes

Let I $=\int\frac{3(3+t)+4}{\sqrt{t}}dt$

$$=\int\frac{9+3t+4}{\sqrt{t}}dt$$

$$=\int\frac{13+3t}{\sqrt{t}}dt$$

$$=\int\left(\frac{13}{\sqrt{t}}+\frac{3t}{\sqrt{t}}\right)dt$$

$$=13\int\frac{dt}{\sqrt{t}}+3\int t^{1/2}dt$$

$$=13\cdot 2\sqrt{t}+3\cdot\frac{t^{3/2}}{\frac{3}{2}}+C$$

$$=26\sqrt{x-3}+2(x-3)^{3/2}+C$$

Integrals Based On

$\int\frac{1}{\mathrm{a}\pm\mathrm{b}\sin^2 x}dx$, $\int\frac{1}{\mathrm{a}\pm\mathrm{b}\cos^2 x}dx$, $\int\frac{1}{\mathrm{a}\sin^2 x\pm\mathrm{b}\cos^2 x}dx$.

Ex. Evaluate $\int \frac{1}{4\sin^2 x - 9\cos^2 x} dx$

Solution:

Let I $= \int \frac{1}{4\sin^2 x - 9\cos^2 x} dx$ **…** (1)

Taking $4\cos^2 x$ common in denominator

$$I = \int \frac{1}{4\cos^2 x} \frac{dx}{\left[\frac{4\sin^2 x}{4\cos^2 x} - \frac{9\cos^2 x}{4\cos^2 x}\right]}$$

$$= \int \frac{dx}{4\cos^2 x\left[\tan^2 x - \frac{9}{4}\right]}$$

$$I = \frac{1}{4}\int \frac{\sec^2 x dx}{\tan^2 x - \left[\frac{3}{2}\right]^2}$$

Put $\tan x = t$ Differentiating both side w.r.t **x**

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}(t) \qquad \ldots\left(\frac{d}{dx}\tan x = \sec^2 x\right)$$

$$\sec^2 x = \frac{dt}{dx}$$

$$\sec^2 x dx = dt$$

$\therefore$ Equation (1) becomes

$$I = \frac{1}{4}\int \frac{dt}{t^2 - \left(\frac{3}{2}\right)^2} dt$$

By using Formula

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + \text{c} \quad \text{here } a = \frac{3}{2}$$

$$= \frac{1}{4} \times \frac{1}{2 \times \frac{3}{2}} \log\left(\frac{t - \frac{3}{2}}{t + \frac{3}{2}}\right) + c$$

$$= \frac{1}{12}\log\left(\frac{\tan x - \frac{3}{2}}{\tan x + \frac{3}{2}}\right) + c \qquad \text{….. Put back } t = \tan x$$

HW Ex. Evaluate $\int \frac{dx}{2-3\cos^2 x}$

Solution:

Let I $= \int \frac{dx}{2-3\cos^2 x}$

Divide numerator and denominator by $\cos^2 x$

$$I = \int \frac{dx/\cos^2 x}{\frac{2-3\cos^2 x}{\cos^2 x}} = \int \frac{\sec^2 x dx}{\frac{2}{\cos^2 x} - \frac{3\cos^2 x}{\cos^2 x}}$$

$$= \int \frac{\sec^2 x dx}{2\sec^2 x - 3}$$

Put $\sec^2 x = 1 + \tan^2 x$

$$= \int \frac{\sec^2 x dx}{2(1+\tan^2 x) - 3}$$

$$= \int \frac{\sec^2 x dx}{2 + 2\tan^2 x - 3}$$

$$= \int \frac{\sec^2 x dx}{2\tan^2 x - 1}$$

?

HW Ex. $\int \frac{dx}{8-7\sin^2 x}$

Integration by Partial Fraction

Ex. $\int \frac{1}{x^2+4x-5}dx$

Solution:

Let I $= \int \frac{1}{x^2+4x-5}dx$

$= \int \frac{1}{(x+5)(x-1)}dx$

Solve this by partial fraction

$\frac{1}{(x+5)(x-1)} = \frac{A}{(x+5)} + \frac{B}{(x-1)}$ ….. (1)

$1 = A(x-1)+B(x+5)$ ….. (2)

Put x=1 in equation (2)

$1 = A(0)+B(6)$

$\therefore$ B $= \frac{1}{6}$

Put x=-5 in equation (2)

$1 = A(-5-1)+B(-5+5)$

$= A(-6)$

$\therefore$ A $= -\frac{1}{6}$

$\therefore$ Equation (1) becomes

$\frac{1}{(x+5)(x-1)} = \frac{-1}{6(x+5)} + \frac{B}{6(x-1)}$

$\int \frac{1}{(x+5)(x-1)}dx = \frac{1}{6}\int \frac{1}{(x+5)}dx + \frac{1}{6}\int \frac{1}{(x-1)}dx$

$= -\frac{1}{6}\frac{\log|x+5|}{\frac{d}{dx}(x+5)} + \frac{1}{6}\frac{\log|x-1|}{\frac{d}{dx}(x-1)} + c$ …… $\int \frac{1}{f(x)}dx = \frac{\log xf(x)}{\frac{d}{dx}[f(x)]}$

$= -\frac{1}{6}\log|x+5| + \frac{1}{6}\log|x-1| + c$

Ex. $\int \frac{(x+1)dx}{(x+2)(x+3)}$

Let I **=** $\int \frac{(x+1)dx}{(x+2)(x+3)}$

Solve this by partial fraction

$$\frac{(x+1)}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)} \quad \text{..... (1)}$$

$$\frac{(x+1)}{(x+2)(x+3)} = \frac{A(x+3)+B(x+2)}{(x+2)(x+3)}$$

$$(x+1) = A(x+3)+B(x+2) \quad \text{..... (2)}$$

Put x=-3 in equation (2)

-3+1 = $A(-3+3)+B(-3+2)$

-2 = B (-1)

$\therefore$ B = 2

Put x= -2 in equation (2)

-2+1 = $A(-2+3)+B(-2+2)$

-1 = A (1)

$\therefore$ A = -1

Put values of A and B in Equation (1)

$$\frac{(x+1)}{(x+2)(x+3)} = \frac{-1}{(x+2)} + \frac{2}{(x+3)}$$

$$\int \frac{(x+1)dx}{(x+2)(x+3)} = \int \left[\frac{-1}{(x+2)} + \frac{2}{(x+3)} \right] dx$$

?

HW Ex. $\int \frac{dx}{2x^2 - 5x + 2}$

HW Ex. $\int \frac{\sin x}{(1+\cos x)(2+\cos x)}dx$ Hint: Put cos x = t

HW Ex. $\int \frac{\sec^2 x}{(1+\tan x)(2+\tan x)}dx$ Hint: Put tan x = t

Integral of the form

$\int \frac{dx}{ax^2+bx+c}$; $\int \frac{dx}{\sqrt{ax^2+bx-c}}$

Ex. Evaluate $\int \frac{dx}{x^2 - 10x + 34}$

Let I $= \int \frac{dx}{x^2 - 10x + 34}$

We can not factorize $x^2 - 10x + 34$ so we have to calculate 3rd term

So 3rd term = $(\frac{1}{2} \times \text{coefficient of x})^2 = (\frac{1}{2} \times (-10))^2 = [-5]^2 = 25$

Now add and subtract this 3rd term to get a perfect square

$\therefore$ Equation (1) becomes

$\therefore$ I $= \int \frac{dx}{x^2 - 10x + 25 + 34 - 25} = \int \frac{dx}{(x-5)^2}$

$= \int \frac{dx}{(x-5)^2 + 3^2}$

By using formula

$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left[\frac{x}{a}\right] + c$ Here a = 3

I $= \frac{1}{3} \tan^{-1}\left[\frac{x-5}{3}\right] + c$ ….. **Ans.**

Ex. Evaluate $\int \frac{dx}{\sqrt{4x - x^2}}$

Let I $= \int \frac{dx}{\sqrt{4x - x^2}}$

We can not factorize $4x - x^2$ so we have to calculate 3rd term

So 3rd term = $(\frac{1}{2} \times \text{coefficient of x})^2 = (\frac{1}{2} \times (4))^2 = 4$

Now add and subtract this 3rd term to get a perfect square

$\therefore$ I $= \int \frac{dx}{\sqrt{4 - 4 + 4x - x^2}} = \int \frac{dx}{\sqrt{4 - x^2 + 4x - 4}}$

$\therefore$ I $= \int \frac{dx}{\sqrt{4 - [x^2 - 4x + 4]}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$

?

HW Ex. $\int \frac{dx}{\sqrt{x^2 - 3x + 4}}$

Ex. Evaluate $\int \frac{dx}{\sqrt{x^2 - 3x + 4}}$

Let I $= \int \frac{dx}{\sqrt{x^2 - 3x + 4}}$ dx …. (1)

We can not factorize $x^2 - 3x + 4$ have to calculate 3rd term

So 3rd term = $(\frac{1}{2} \times \text{coefficient of x})^2 = (\frac{1}{2} \times (-3))^2 = \frac{9}{4}$

Now add and subtract this 3rd term to get a perfect square

$\therefore$ Equation (1) becomes

$$\therefore \quad I = \int \frac{dx}{\sqrt{x^2 - 3x + \frac{9}{4} + 4 - \frac{9}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 + \frac{16 - 9}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}}$$

By using formula

$$\int \frac{dx}{x^2 + a^2} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$$

Here x = $\left(x - \frac{3}{2}\right)$ and a= $\frac{\sqrt{7}}{4}$

$$I \quad = \log\left|\left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}\right| + c$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + \frac{9}{4} + \frac{7}{4}}\right| + c$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + \frac{16}{4}}\right| + c$$

$$= \log\left|\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 4}\right| + c \qquad \text{……}\textbf{Ans.}$$

HW Ex. $\int \frac{dx}{\sqrt{x^2 - 5x + 6}}$

Integrals of the Form

(i) $\int \frac{dx}{a+b\cos 2x}$ **(ii)** $\int \frac{dx}{a-b\cos 2x}$ **(iii)** $\int \frac{dx}{a+b\sin 2x}$

(iv) $\int \frac{dx}{a-b\sin 2x}$ **(v)** $\int \frac{dx}{a\cos 2x-b\sin 2x}$

For solving these types of integration by putting t = tan x
Differentiating w.r.t. x, We get

$$\frac{d}{dx}(t) = \frac{d}{dx}(\tan x)$$

$$\therefore \quad \frac{dt}{dx} = \frac{d}{dx}\sec^2 x$$

$$\therefore \quad dx = \frac{dt}{\sec^2 x} = \frac{dt}{1+\tan^2 x} = \frac{dt}{1+t^2} \qquad \ldots.\ t = \tan x$$

Now, as we know

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x} = \frac{1-t^2}{1+t^2}$$

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x} = \frac{2t}{1+t^2}$$

Ex. Evaluate $\int \frac{1}{5+4\cos 2x}dx$

Let I $= \int \frac{1}{5+4\cos 2x}dx$ …. (1)

Put tan x = t
Differentiating W.r.t. x,

$$\sec^2 x = \frac{dt}{dx}$$

$$(1+\tan^2 x)dx = dt$$

$\therefore$ $$dx = \frac{dt}{1+\tan^2 x}$$

$$dx = \frac{dt}{1+t^2 x}, \quad \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}, \quad \cos 2x = \frac{1-t^2}{1+t^2}$$

$\therefore$ Equation (1) becomes

$$I = \int \frac{\frac{dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$I = \int \frac{\frac{dt}{1+t^2}}{\frac{5(1-t^2)+4(1-t^2)}{(1+t^2)}} = \int \frac{dt}{5+5t^2+4-4t^2}$$

$$= \int \frac{dt}{t^2+9}$$

$$I = \int \frac{dt}{t^2+3^2}$$

By Using $\int \frac{dt}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

$$I = \frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right)+c$$

$$I = \frac{1}{3}\tan^{-1}\left(\frac{\tan x}{3}\right)+c$$

Ex. Evaluate $\int \frac{dx}{3-4\sin 2x}$

Let $I = \int \frac{dx}{3-4\sin 2x}$ …. (1)

Put tan x = t Differentiating W.r.t. x,

$\therefore$ $$\frac{d}{dx}(\tan x) = \frac{d}{dx}(t)$$

$$\sec^2 x = \frac{dt}{dx}$$

$\therefore$ $$dx = \frac{dt}{\sec^2 x} = \frac{dt}{1+\tan^2 x} = \frac{dt}{1+t^2}$$

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}, \quad \sin 2x = \frac{2t}{1+t^2}$$

$\therefore$ Equation (1) becomes

$$I = \int \frac{\frac{dt}{1+t^2}}{3-4\left(\frac{2t}{1+t^2}\right)} = \int \frac{\frac{dt}{1+t^2}}{\frac{3(1+t^2)-8t}{(1+t^2)}}$$

$$I = \int \frac{dt}{3+3t^2-8t} = \frac{1}{3}\int \frac{dt}{t^2-\frac{8}{3}t+1} \quad \text{.......(2)}$$

We can not factorize $t^2-\frac{8}{3}t+1$ so we will calculate 3rd term

So 3rd term = $\left(\frac{1}{2}\times \text{coefficient of x}\right)^2 = \left[\frac{1}{2}\times\left(\frac{-8}{3}\right)\right]^2 = \left[-\frac{4}{3}\right]^2 = \frac{16}{9}$

$\therefore$ Equation (2) becomes, (Adjusting 3rd term)

$$I = \frac{1}{3}\int \frac{dt}{t^2-\frac{8}{3}t+\frac{16}{9}+1-\frac{16}{9}}$$

$$= \frac{1}{3}\int \frac{dt}{\left(t-\frac{4}{3}\right)^2+\frac{9-16}{9}} = \frac{1}{3}\int \frac{dt}{\left(t-\frac{4}{3}\right)^2-\frac{7}{9}}$$

$$= \frac{1}{3}\int \frac{dt}{\left(t-\frac{4}{3}\right)^2-\left(\frac{\sqrt{7}}{3}\right)^2}$$

By Using $\int \frac{1}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

$\therefore$ Here $a = \frac{\sqrt{7}}{3}$

$$I = \frac{1}{3} \times \frac{1}{2 \times \frac{\sqrt{7}}{3}} \log \left| \frac{\left(t - \frac{4}{3} - \frac{\sqrt{7}}{3}\right)}{\left(t - \frac{4}{3} + \frac{\sqrt{7}}{3}\right)} \right| + c$$

$$I = \frac{1}{2\sqrt{7}} \log \left| \frac{3t - 4 - \sqrt{7}}{3t - 4 + \sqrt{7}} \right| + c$$

$$I = \frac{1}{2\sqrt{7}} \log \left| \frac{3\tan x - 4 - \sqrt{7}}{3\tan x - 4 + \sqrt{7}} \right| + c \qquad \ldots.\ t = \tan x$$

Integrals of the Form

(i) $\int \frac{dx}{a + b\cos x}$ **(ii)** $\int \frac{dx}{a - b\cos x}$ **(iii)** $\int \frac{dx}{a + b\sin x}$

(iv) $\int \frac{dx}{a - b\sin x}$ **(v)** $\int \frac{dx}{a\cos x + b\sin x}$

For intergrating expressions of these types, we use the substitution $\tan\frac{x}{2} = t$, then,

Differentiating w.r.t. x, We get

$$\sin x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{2t}{1 + t^2} \qquad \cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{1 - t^2}{1 + t^2}$$

Also differentiating, $\tan\frac{x}{2} = t$, we get

$$\frac{d}{dx}\left(\tan\frac{x}{2}\right) = \frac{d}{dx}(t) \qquad \therefore \quad \sec^2\frac{x}{2} \cdot \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{dt}{dx}$$

$$\frac{1}{2}\sec^2\frac{x}{2}.dx = dt \Rightarrow dx = \frac{2dt}{1 + \tan^2\left(\frac{x}{2}\right)} \Rightarrow dx = \frac{2t}{1 + t^2}$$

This substitution transforms above expressions to algebraic from that can be easily integrated.

e.g. $\therefore \quad I = \int \frac{dx}{a + b\cos x} = \int \frac{\frac{2dt}{1 + t^2}}{a + b\frac{1 - t^2}{1 + t^2}} = \int \frac{2dt}{a(1 + t^2) + b(1 - t^2)}$

$$= \int \frac{2dt}{(a+b)+(a-b)t^2}$$

$$= \frac{2}{(a-b)}\int \frac{dt}{\left(\frac{a+b}{a-b}\right)+t^2} \qquad \textbf{Form } \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$I = \frac{2}{a-b}\cdot\frac{1}{\sqrt{\frac{a+b}{a-b}}}\tan^{-1}\left\{\frac{t}{\sqrt{\frac{a+b}{a-b}}}\right\}+c$$

$$I = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\sqrt{\frac{a+b}{a-b}}\cdot t\right\}+c$$

$$I = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\sqrt{\frac{a+b}{a-b}}\cdot \tan\frac{x}{2}\right\}+c$$

Ex. $\int \frac{1}{1+3\cos x}dx$

Let I $= \int \frac{1}{1+3\cos x}dx$ …. (1)

Substantiation $\tan\frac{x}{2} = t \quad \therefore \quad \cos x = \frac{1-t^2}{1+t^2} \quad = \quad dx = \frac{2dt}{1+t^2}$

$\therefore$ Equation (1) becomes

$$I = \int \frac{2dt}{(1+t^2)\left[1+\frac{3(1-t^2)}{1+t^2}\right]} = \int \frac{2dt}{(1+t^2)\left[\frac{1+t^2+3-3t^2}{1+t^2}\right]}$$

$$= \int \frac{2dt}{4-2t^2} = \frac{2}{2}\int \frac{2dt}{2-t^2} = \int \frac{dt}{(\sqrt{2})^2-t^2}$$

$$I = \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right|+c \qquad \ldots\ldots \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$$

Now put back $\tan\frac{x}{2} = t$

$$I = \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\tan\frac{x}{2}}{\sqrt{2}-\tan\frac{x}{2}}\right|+c \qquad \ldots\ldots\textbf{Ans.}$$

Ex. $\int \frac{1}{5+4\cos x}dx$

Let I $= \int \frac{1}{5+4\cos x}dx$ …. (1)

Substituting $\tan\frac{x}{2} = t \quad \Rightarrow \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}$

$\therefore$ Equation (1) becomes

I $= \int \frac{\frac{2dt}{1+t^2}}{5+4\frac{(1-t^2)}{(1+t^2)}}dx$

?

HW Ex. 1. $\int \frac{1}{4+5\cos x}dx$

HW Ex. 2. $\int \frac{1}{5-4\sin x}dx$

Ex. $\int \frac{1}{2+\cos x-\sin x}dx$

Let I $= \int \frac{1}{2+\cos x-\sin x}dx$ …. (1)

Substituting $\tan\frac{x}{2}=t$, $dx = \frac{2dt}{1+t^2}$

$\therefore$ $\cos x=\frac{1-t^2}{1+t^2}$ **,** $\sin x=\frac{2t}{1+t^2}$

$\therefore$ Equation (1) becomes

$$I = \int \frac{2dt}{\left(1+t^2\right)\left[2+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}\right]} = \int \frac{2dt}{\left(1+t^2\right)\left[\frac{2\left(1+t^2\right)+\left(1-t^2\right)-2t}{\left(1+t^2\right)}\right]}$$

$$= \int \frac{2dt}{2+2t^2+1-t^2-2t} = \int \frac{2dt}{t^2-2t+3}$$

Now, calculate 3rd term here

So 3rd term = $\left(\frac{1}{2} \times \text{coefficient of x}\right)^2 = \left[\frac{1}{2}\times(-2)\right]^2 = 1$

$$I = \int \frac{2dt}{t^2-2t+1+3-1} = \int \frac{2dt}{t^2-2t+1+2}$$

$$I = 2\int \frac{dt}{(t+1)^2+\left(\sqrt{2}\right)^2}$$

Form $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$\therefore \quad I = 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{t-1}{\sqrt{2}}\right\}+c$$

Now put back $\tan\frac{x}{2} = t$

$$I = \sqrt{2}\tan^{-1}\left\{\frac{\tan\frac{x}{2}-1}{\sqrt{2}}\right\}+c \qquad \ldots\ldots\textbf{Ans.}$$

H.W. Ex $\int \frac{1}{3\cos x - 2}$

Integration by Parts

$$\int (u\cdot v)dx = u\int v dx - \int\left[\int v dx \cdot \frac{d}{dx}(u)\right]dx$$

But we have to chose u and v i.e. u is first function and v is second function by using LIATE rule.

L = Logarithmic Function e.g. $\log x, \log(1+x)$ etc.

L = Inverse trigonometric Function e.g. $\sin^{-1} x, \tan^{-1} x$ etc.

L = Algebraic Function e.g. x, x^2, x^8 etc.

L = Trigonometric Function e.g. $\sin x, \cos x$ etc.

L = Exponential Function e.g. e^x, a^x etc.

Ex. Evaluate $\int xe^x dx$

Let I = $\int xe^x dx$

Here x is an algebraic Function and e^x is an Exponential Function

$\therefore$ Integration by parts

By LIATE rule, u = v; v = e^x

Formula

$$\int (u \cdot v) dx = u\int vdx - \int\left[\int vdx \cdot \frac{d}{dx}(u)\right]dx$$

$$\int xe^x dx = x\int e^x dx - \int\left[\int e^x dx \cdot \frac{d}{dx} x\right]dx$$

$$= xe^x - \int \lfloor e^x \cdot 1 \rfloor dx \qquad \text{....}\left(\because \int e^x dx = \lfloor e^x \rfloor\right)$$

$$= xe^x - e^x + c$$

$$\text{I} = e^x(x-1) + c$$

Ex. Evaluate $\int x^{2007} \log x dx$

Let I = $\int x^{2007} \log x dx$

Here logx is an Logarithmic Function and x^{2007} is an algebraic function

$\therefore$ Integration by parts

By LIATE rule, u = logx; v = x^{2007}

Formula

$$\int (u \cdot v) dx = u\int vdx - \int\left[\int vdx \cdot \frac{d}{dx}(u)\right]dx$$

$$= \log x\int x^{2007} dx - \int\left[\int x^{2007} dx \cdot \frac{d}{dx}(\log x)\right]dx$$

$$= \log x \frac{x^{2007+1}}{2007+1} - \int\left[\frac{x^{2007+1}}{2007+1} \cdot \frac{1}{x}\right]dx$$

$$= \log x \frac{x^{2008}}{2008} - \int\left[\frac{x^{2008}}{2008} \cdot \frac{1}{x}\right]dx$$

$$= \frac{x^{2008}}{2008}\log x - \int\left[\frac{x^{2007}}{2008}\right]dx$$

$$= \frac{x^{2008}}{2008}\log x - \frac{1}{2008}\int x^{2008}dx$$

$$= \frac{x^{2008}}{2008}\log x - \frac{x^{2008}}{(2008)^2} + c \qquad \ldots\ldots\left[\int x^n dx = \frac{x^{n+1}}{n+1}\right]$$

$$= \frac{x^{2008}}{2008}\left[\log x - \frac{1}{2008}\right] + c$$

HW Ex. **1.** $\int x \cdot \cos x dx$ **2.** $\int \log x dx$

Ex. Evaluate $\int \tan^{-1} x dx$

Let I = $\int \tan^{-1} x dx$

We do not have formula for $\int \tan^{-1} x dx$

So we take second function as unity i.e. 1.

$\therefore$ I = $\int \tan^{-1} x \cdot 1 dx$

By LIATE rule first function

u = $\tan^{-1} x$, since $\tan^{-1} x$ is inverse trigonometric function and

v = 1, since 1 is algebraic function

I = $\int \tan^{-1} x \cdot 1 dx$

Formula of integrating by parts

$$\int (u \cdot v)dx = u\int vdx - \int\left[\int vdx \cdot \frac{d}{dx}(u)\right]dx$$

$$\int \tan^{-1} x \cdot 1dx = \tan^{-1} x\int 1dx - \int\left[\int 1dx \cdot \frac{d}{dx}\left(\tan^{-1} x\right)\right]dx$$

$$= \tan^{-1} x[x] - \int\left[x \cdot \frac{1}{1+x^2}\right]dx \qquad \ldots\int e^x dx = x$$

$$f'\{x\}$$

$$\downarrow$$

$$= \tan^{-1} x[x] - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$\uparrow$$

$f\{x\}$ … Multiply and divide by 2

$$= \quad \tan^{-1} x[x] - \frac{1}{2}\log\left|1+x^2\right| + c \qquad \ldots \frac{f'(x)}{f(x)}dx = \log\left|f(x)\right| + c$$

Ex. Evaluate $\int \cot^{-1} x dx$

Let I = $\int \cot^{-1} x dx$ = $\int \cot^{-1} x \cdot 1 dx$

$\downarrow$ $\uparrow$

I A

This is a combination of inverse and algebraic functions

By LIATE rule

u $= \cot^{-1} x$, and v = 1

Formula of integrating by parts

$$\int (u \cdot v) dx = u\int v dx - \int\left[\int v dx \cdot \frac{d}{dx}(u)\right]dx$$

$$\int \cot^{-1} x \cdot 1 dx = \cot^{-1} x \int 1 dx - \int\left[\int 1 dx \cdot \frac{d}{dx}\left(\cot^{-1} x\right)\right]dx + c$$

?

HW.Ex. 1. $\int x \tan^{-1} x dx$ **.**

HW.Ex 2. $\int x^2 e^x dx$

HW.Ex 3. $\int x^2 \cos 2x dx$

HW.Ex 4. $\int e^{4x} \cos 3x dx$

Ex. Evaluate $\int x^2 \sin x dx$

Let I = $\int x^2 \sin x dx$

By LIATE rule x^2 algebraic function and $\sin x$ is trigonometric function

So u $= x^2$, and v = $\sin x$

Formula of integrating by parts

$$\int (u \cdot v) dx = u \int v dx - \int \left[\int v dx \cdot \frac{d}{dx}(u) \right] dx$$

$$\int x^2 \sin x dx = x^2 \int \sin x dx - \int \left[\int \sin x dx \cdot \frac{d}{dx}(x^2) \right] dx$$

?

H.W.Ex. 1. $\int \sin^{-1}\sqrt{x}\,dx$