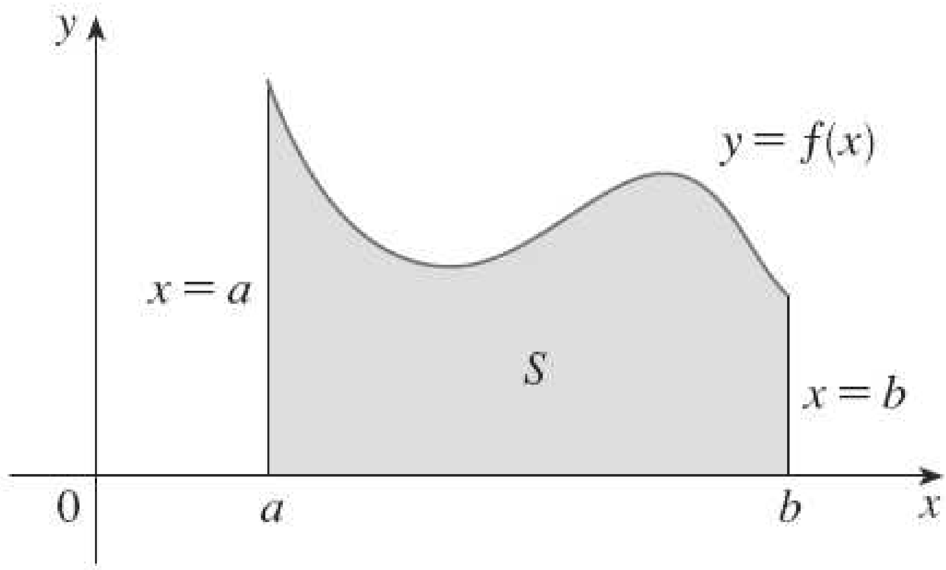
Application Of Definite Integrals:-

1. **Area under the Curve:**
   Consider the curve $y = f(x)$, then the area under the curve $y = f(x)$ and the ordinate $x = a$ and $x = b$ and the x axis is given by

$$A = \int_{x=a}^{x=b} y\,dx \qquad \text{OR} \qquad A = \int_{x=a}^{x=b} f(x)\,dx\,.$$

The area under the curve $x = g(y)$,the ordinate $y = c$ and $y = d$ and x axis is

$$A = \int_{y=d}^{y=c} x\,dy \qquad \text{OR} \qquad A = \int_{y=d}^{y=c} g(y)\,dy$$



Ex.1 Obtain the area between line $y = 8x$, x axis and ordinates at $x = 2$ and $x = 6$
Soln.:

Area bounded = $\int_{x=2}^{x=6} y\,dx = \int_{x=2}^{x=6} 8x\,dx$

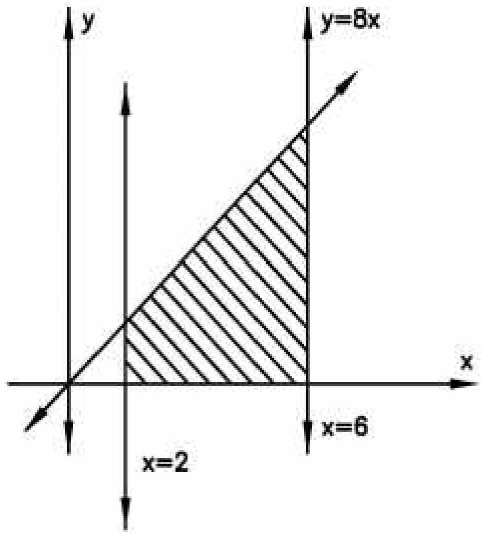
$$= 8\left[\frac{x^2}{2}\right]_2^6 \qquad \ldots \int_a^b x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

$$= 4\left[x^2\right]_2^6$$

$$= 4\left[6^2 - 2^2\right]$$

= 4 [36-4]

$= 4[32]$
$=128$ Sq.units



Ex.2: Find the area bounded by the curve $y = x^3$, x axis and the coordinate. $x = 1, x = 3$

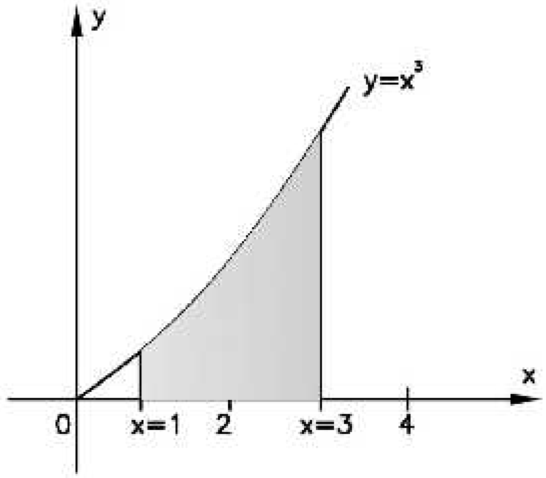
Soln.: The area bounded by the curve $y = x^3$, x axis and the coordinate. $x = 1, x = 3$

$\therefore$ The required area $A = \int_1^3 y.dx$

$$= \int_1^3 x^3.dx = \left[\frac{x^{3+1}}{3+1}\right]_1^3 = \left[\frac{x^4}{4}\right]_1^3$$

$$= \frac{1}{4}\left[x^4\right]_1^3 = \frac{1}{4}\left[3^4 - 1^4\right]$$

$$= \frac{1}{4}[81-1] = \frac{1}{4}[80] = 20unit^2$$



Ex.3: Find the area of the region bounded by the curve $y = 4x^2$, x axis and the lines. $x = 1$ and $x = 2$.

Soln.: The required area is as shown in Fig.

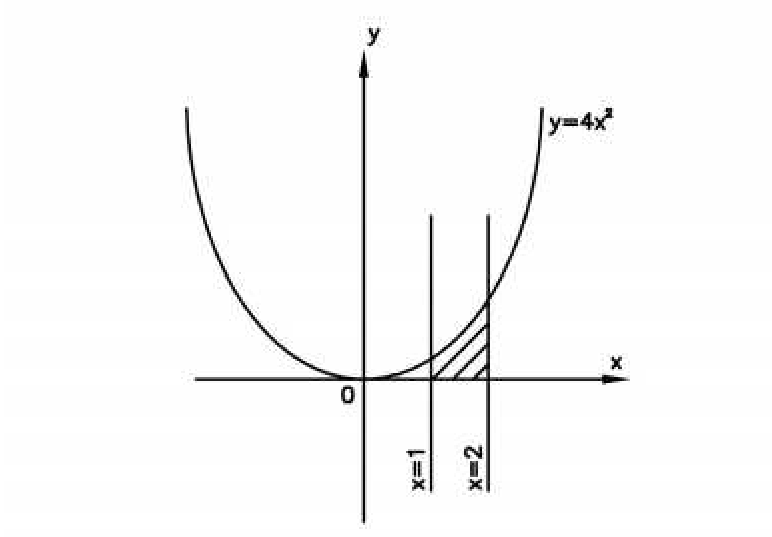
$$\therefore \text{Required area } A = \int_1^2 y.dx = \int_1^2 4x^2.dx$$

$$= 4\int_1^2 x^2.dx \quad = 4\left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{4}{3}\left[(2)^3 - (1)^3\right]$$

$$= \frac{4}{3}(8-1) = \frac{4}{3}(7)$$

$$= \frac{28}{3} \text{ square units.}$$



Ex.4: Find the area bounded by $y = 4x - x^2$, meeting the x axis and the ordinates $x = 1$, $x = 3$.

Soln.: here given curve $y = 4x - x^2$ is parabola meeting x axis at the $(0,0)$ and $(4,0)$ as in the fig.

$$\therefore \text{Required area} = \int_{x=1}^{x=3} y.dx = \int_{1}^{3} \left(4x - x^2\right) dx$$

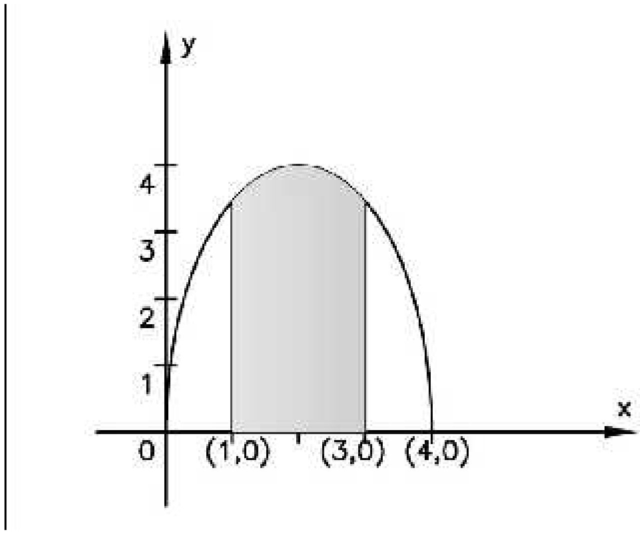
$$= \left[4\frac{x^2}{2} - \frac{x^3}{3}\right]_1^3$$

$$= 2\left(3^2 - 1^2\right) - \frac{1}{3}\left(3^3 - 1^3\right)$$

$$= 2(9-1) - \frac{1}{3}(27-1)$$

$$\therefore \quad \text{Area} = 16 - \frac{26}{3}$$

$$= \frac{22}{3} \text{ sq.units.}$$



Ex.5: Find the area enclosed by curve $y = 4 - x^2$ and the lines $x = 0$, $x = 2$, $y = 0$

Soln.: Given curve is the parabola with vertex here $(0,4)$ meeting x axis at $(2,0)$ $(-2,0)$ as in the

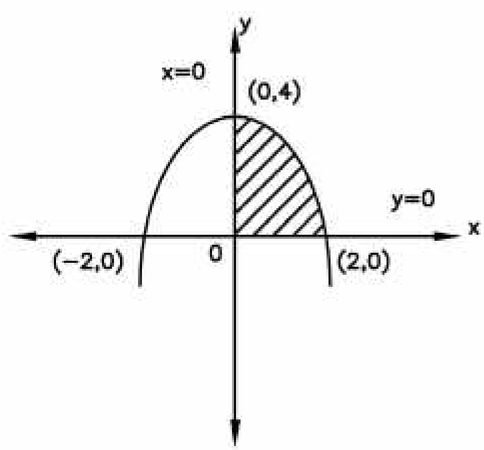


fig.

$$\therefore \text{Required area } = \int_{x=0}^{x=2} y.dx = \int_{1}^{2} \left(4 - x^2\right) dx$$

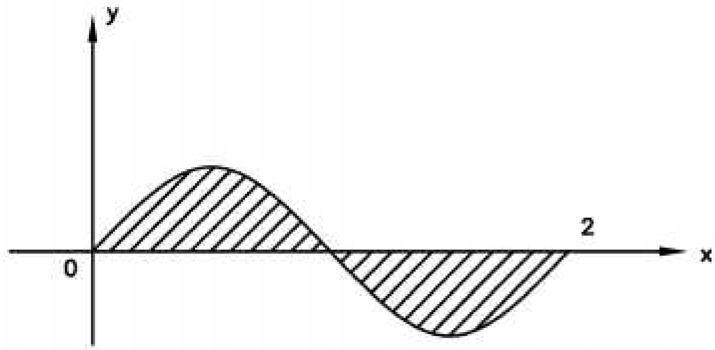
$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 4(2-0) - \frac{1}{3}\left(2^3 - 0\right)$$

$$= 8 - \frac{8}{3}$$

$$= \frac{16}{3} \text{ sq.units.}$$

Ex.6: Find the area under the curve $y = \sin x$ from $x = 0$ to $x = 2®$



Soln.: Fig shows the graph $y = \sin x$
The are from 0 to ® lies in the 1st quadrant and area from ® to 2® is below the axis and it is in the IV^{th} quadrant.

$$A = 2\int_0^{®} y.dx = 2\int_0^{®} \sin x.dx$$

$$= [-2\cos x]_0^{®} \qquad \text{….As } \int_a^b \sin x.dx = [-\cos]_a^b$$

Ex.7: Find the area bounded by curve $y = 1 + x^3 + 2\sin x$, the x-axis and ordinates $x = 0$, $x = ®$
Soln.:

$$\therefore \text{ Required area } = \int_{x=0}^{x=®} y.dx = \int_{x=0}^{x=®} \left(1 + x^3 + 2\sin x\right)dx$$

$$= \int_0^{®} dx + \int_0^{®} x^3 dx + 2\int_0^{®} \sin x dx$$

Ex.8: Find the area between the parabola $y = 4x - x^2$ and the x-axis

Soln.: The equation is $y = 4x - x^2$

When $y = 0$ $x = 0$

When $y = 0$ $4x - x^2 = 0$

$x(4 - x) = 0$

$\therefore$ $x = 0$ or $x = 4$

$$A = \int_0^4 y.dx = \int_0^4 \left(4x - x^2\right).dx$$

Ex.9: Find the area enclosed by curve $y = 4 - x^2$ and the x-axis

Soln.: The equation of curve is $y = 4x - x^2$

When $y = 0$

$0 = 4 - x^2$

$\therefore$ $x^2 = 4$

$\therefore$ $x = \pm 2$

$\therefore$ The point of inter –section of parabola with x-axis is $(-2,0)$ and $(2,0)$

$$\therefore A = \int_{-2}^{2} y.dx = \int_{-2}^{2} \left(4 - x^2\right)dx$$

As $f(x) = 4 - x^2$ is an even function

$$= = 2\int_0^2 \left(4 - x^2\right)dx \qquad \ldots \left[\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\right]$$

Ex.10: Find the area enclosed between the curve $y = 3x - 2 - x^2$ and the x-axis

Soln.: Given equation of curve

$$y = 3x - 2 - x^2$$

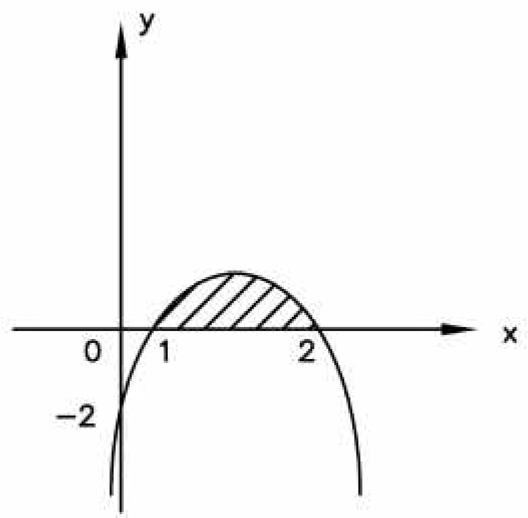
x	0	1	2	3
y	-2	0	0	-2

Area $= \int y\,dx = \int_1^2 \left(3x - 2 - x^2\right)dx$

$$= \left(3\frac{x^2}{2} - 2x - \frac{x^3}{3}\right)_1^2$$

$$= \left[\frac{3}{2}(2)^2 - 2(2) - \frac{(2)^3}{3}\right] - \left[\frac{3}{2}(1)^2 - 2(1) - \frac{(1)^3}{3}\right]$$

$$= \frac{3}{2}\times 4 - 4 - \frac{8}{3} - \frac{3}{2} + 2 + \frac{1}{3}$$



Ex.11: Find the area of the loop of the curve $y^2 = x^2(1-x)$

Soln.:

Given equation of curve is $y^2 = x^2(1-x)$

Putting $y = 0$ in above equation of the curve

$\therefore\ 0 = x^2(1-x)$

$\therefore\quad x^2 = 0 \quad \text{or} \quad (1-x) = 0$

$\therefore\quad x = 0 \quad \text{or} \quad x = 1$

$\therefore$ Points where the loop cuts x-axis $(0,0)$ and $(1,0)$

$$\therefore\ A = \int_0^1 y.dx = \int_0^1 x\sqrt{1-x}\,dx$$

$\because\ y^2 = x^2(1-x)$ taking square root on both sides $y = x\sqrt{1-x}dx$

$$= \int_0^1 (1-x)\sqrt{1-(1-x)}\,dx \qquad \ldots\left[\int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 (1-x)\sqrt{1-1+x}\,dx$$

$$= \int_0^1 (1-x)\sqrt{1-1+x}\,dx$$

Ex.12: Find the area of the circle $x^2 + y^2 = 25$ using integration.

Soln.: Given circle $x^2 + y^2 = 25$, is with centre $(0,0)$ and radius 5.

$$y^2 = 25 - x^2$$

Now taking square root on both sides

$$y = \sqrt{25 - x^2}$$

$\therefore$ Required area = 4 x area in 1st quadrant

$$\therefore \text{Required area } = 4\int_{x=0}^{x=5} y.dx = 4\int_{1}^{5}\sqrt{25-x^2}\,dx$$

$$= 4\int_{1}^{5}\sqrt{5^2-x^2}\,dx$$

By using formula

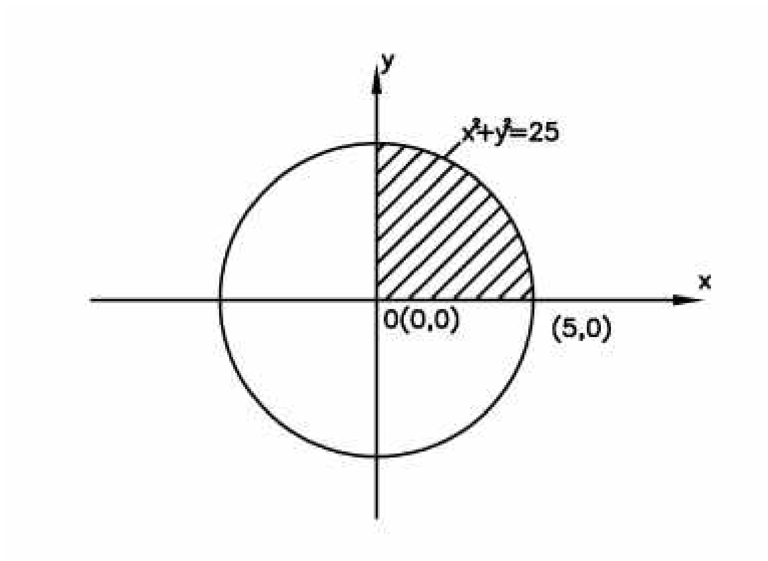
$$\int_{c}^{d}\sqrt{a^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{c}^{d}$$

$$= 4\left[\frac{x}{2}\sqrt{5^2-x^2}+\frac{5^2}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_{0}^{5} \qquad \ldots..(\because a=5)$$

$$=4\left[\frac{5}{2}\sqrt{5^2-5^2}+\frac{5^2}{2}\sin^{-1}\left(\frac{5}{5}\right)-\left(\frac{0}{2}\sqrt{5^2-0^2}+\frac{5^2}{2}\sin^{-1}\left(\frac{0}{5}\right)\right)\right]$$

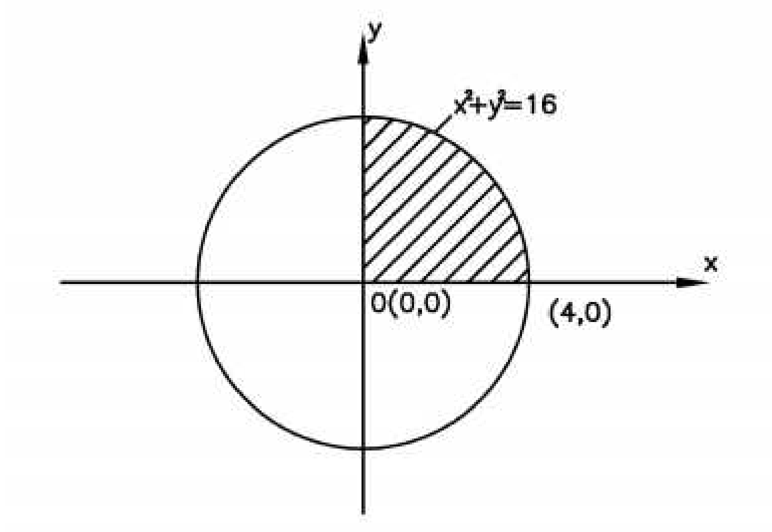
$$= 4\left[0+\frac{25}{2}\sin^{-1}(1)-0\right]=4\left[\frac{25}{2}.\frac{®}{2}\right] \qquad \ldots\sin^{-1}(1)=\frac{®}{2}$$

$$= 25® \text{ sq.units}$$



Ex.13: Find the area of the circle $x^2+y^2=16$ using integration.

Soln.: Given circle $x^2 + y^2 = 16$, is with centre $(0,0)$ and radius 5.



$$y^2 = 16 - x^2$$

Now taking square root on both sides

$$y = \sqrt{16 - x^2}$$

$\therefore$ Required area = 4 x area in 1st quadrant

$$\therefore \text{Required area } = 4\int_{x=0}^{x=4} y.dx = 4\int_{1}^{4}\sqrt{16 - x^2}\,dx$$

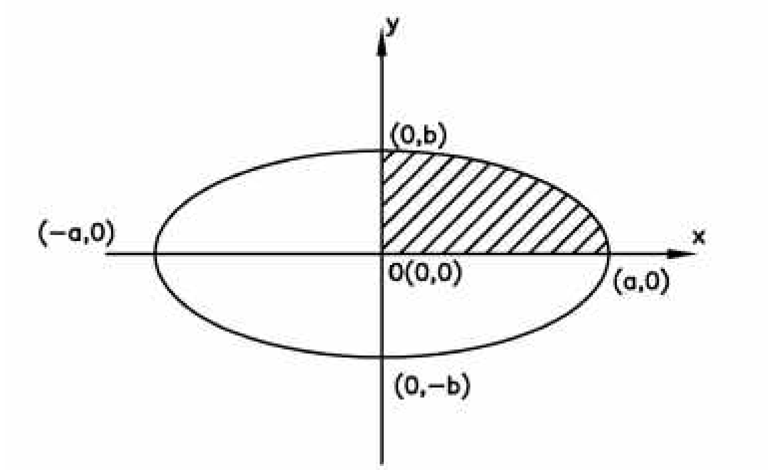
$$= 4\int_{1}^{4}\sqrt{4^2 - x^2}\,dx$$

By using formula

$$\int_{c}^{d}\sqrt{a^2 - x^2}\,dx \quad = \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-2}\left(\frac{x}{a}\right)\right]_{c}^{d}$$

Ex.14: Find the area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by using integration method.

Soln.: Given curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, curve is symmemical abount both the axis.



$$\therefore \text{ Required Area} = 4\int_0^a y\,dx$$

Here $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Now $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

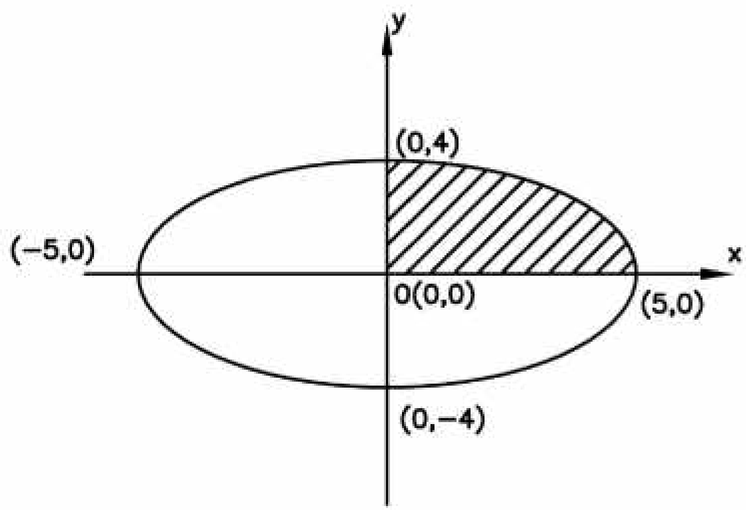
$$\therefore \quad y^2 = b^2\left[1-\frac{x^2}{a^2}\right]$$

$$\therefore \quad y = b\sqrt{1-\frac{x^2}{a^2}}$$

$$\therefore \quad \text{Area} = 4\int_0^a b\sqrt{\frac{a^2-x^2}{a^2}}$$

Home work

Ex.15: Find the area of ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ by using integration method.



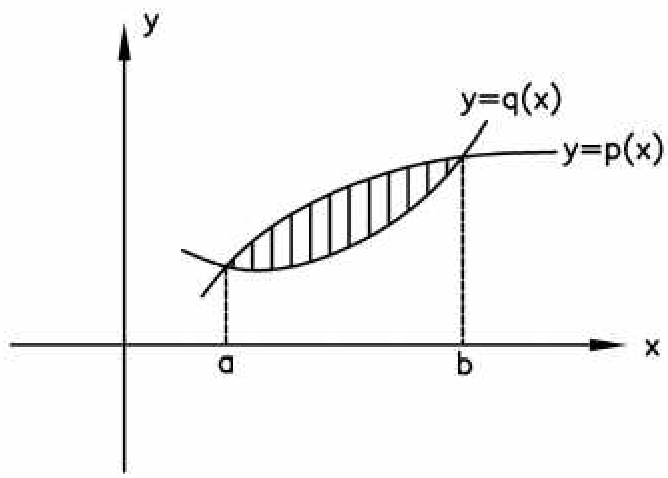
2.Area between Two curves

Let $y = p(x)$ and $y = q(x)$ be the two curvea. As shown in fig.

The area between two curves $y = p(x)$ and $y = q(x)$ is given as,

$$A = \int_a^b p(x)dx - \int_a^b q(x)dx = A_1 - A_2$$

$$= \int_a^b [p(x) - q(x)]dx$$



Ex.16: Find the area between $y = x^2$ and the line $y = x$

Soln.: The given curve $y = x^2$, is parabola opeing upward with vertex at origin $(0,0)$.

The line $y = x$ is passing through origin having slope =1

Two curves intersect

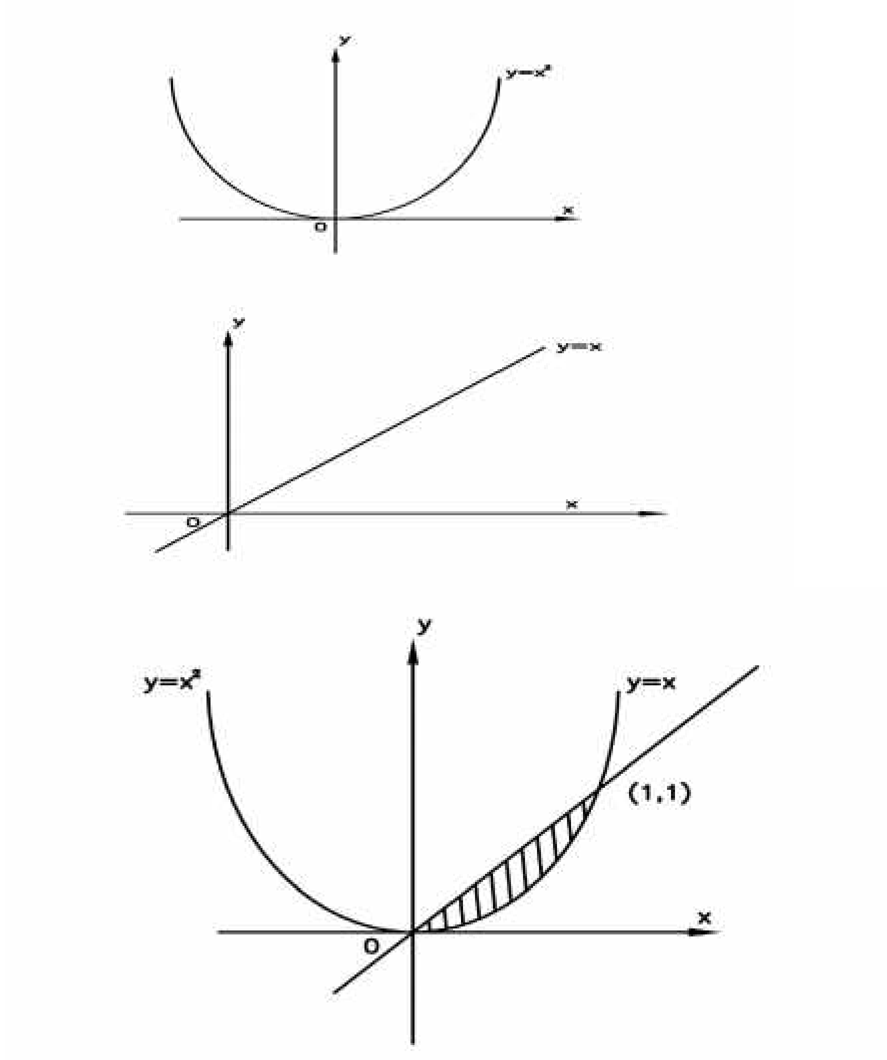
$y = x^2$ and $y = x$

Now, put $y = x^2$ in $y = x$

$\Rightarrow \quad x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow \quad x(x-1) = 0 \quad \Rightarrow \quad x = 0,\ x = 1$

$\therefore \quad y = 0,\ y = 1$



$\therefore$ Curves intersect at the origin and the point $(1,1)$

Required area $= A_1 - A_2 = \int_0^1 y_1 dx - \int_0^1 y_2 dx = \int_0^1 x dx - \int_0^1 x^2 dx$

$$= \left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \text{ sq.units.}$$

Ex.17: Find the area enclosed by $y^2 = 8x$ and the line $x = 2$

Soln.: The required area is bounded by parabola $y^2 = 8x$ and the line $x = 2$ (Parallel to y-axis) as shown in fig.

Line $x = 2$ intersect parabola $y^2 = 8x$ (Symmetric about x-axis)

To find the points of intersection put $x = 2$ in $y^2 = 8x$

$\therefore \quad y^2 = 16 \Rightarrow y = \pm 4$

$\therefore$ Points of intersection are $(2,4)$ $(2,-4)$

$\therefore$ Required area = 2 x area above x-axis

$$= 2\int_{x=0}^{x=2} y.dx = 2\int_0^2 \sqrt{8x}dx = 2\sqrt{8}\int_0^2 x^{1/2}$$

$$= 2\sqrt{8}\left[\frac{x^{3/2}}{3/2}\right] = 2.\frac{2}{3}\sqrt{8}\left[2^{3/2} - 0\right]$$

$$= \frac{4}{3}\sqrt{8}\left(2^3\right)^{1/2} = \frac{4}{3}\sqrt{8}.\sqrt{8}$$

$$= \frac{4}{3}\sqrt{64} = \frac{4}{3}\times 8$$

$$= \frac{32}{3} \text{ sq.units}$$

Ex.18: Find the area bounded by the curve $y^2 = 4x$ and $x^2 = 4y$

Soln.: The required area is area enclosed between the two parabolas $y^2 = 4x$ and $x^2 = 4y$ both intersecting at the points $(0,0)$ $(4,4)$

Now $y^2 = 4x$

Squaring both the sides

$\therefore \quad y^4 = 4^2 \cdot x^2$

$y^4 = 4^2 \cdot 4y \quad \ldots.(\because x^2 = 4y)$

$y^4 = 4^3 y$

$y^4 - 4^3 y = 0$

$y\left(y^3 - 4^3\right) = 0$

$\therefore \quad y = 0, y = 4$ for $y = 4, y^2 = 4x$

$\therefore \quad 4x = 4^2$

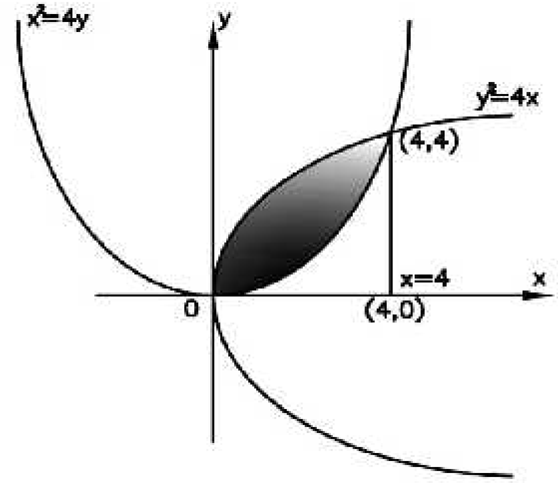
$\therefore \quad x = 4$

Therefore required area $A = A_1 - A_2$

Where $A_1 =$ area bounded by $y^2 = 4x$ and ordinate $x = 4$

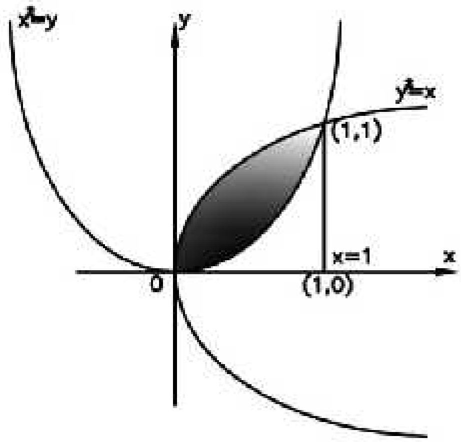
$A_2 =$ area bounded by $x^2 = 4y$ and ordinate $x = 4$

$\therefore$ Required area $= \int_{x=0}^{x=4} y.dx - \int_{x=0}^{x=4} y.dx = \int_{x=0}^{x=4} \sqrt{4} \cdot x^{1/2} dx - \int_{x=0}^{x=4} \frac{x^2}{4} dx$



Home work

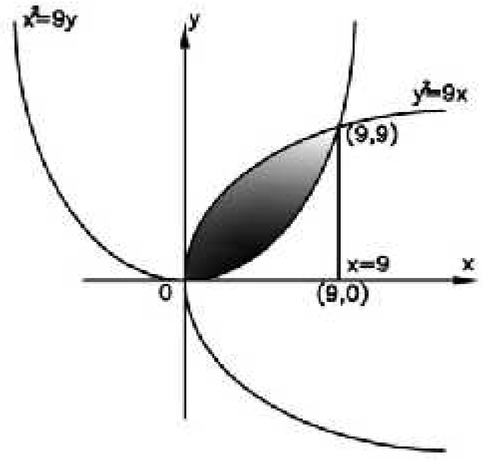
Ex.19: Find the area enclosed by the two parabolas $y^2 = x$ and $x^2 = y$



Ex.20: Find the area bounded between two parabolas $y^2 = 9x$ and $x^2 = 9y$

Soln.: The required area is the area enclosed between the two parabolas

$y^2 = 9x$ and $x^2 = 9y$ both intersecting at the points $(0,0)$ $(9,9)$



Ex.20: Find the area between the parabolas $y = x^2 + 3$ and line $y = x + 3$

Soln.: The required area is the area enclosed between the two parabolas

Given equation of curve

First we will find the ordinates of x and y as follows

x	-2	-1	0	1	2
y	7	4	3	4	7

By using these ordinates plot the curve as shown in fig.

To find points of intersection of the curves

$y = x^2 + 3$ And $y = x + 3$

Putting $y = x + 3$ in $y = x^2 + 3$

$\therefore \quad x + 3 = x^2 + 3$

$\therefore \quad x^2 - x = 0 \qquad \ldots\ldots x(x-1) = 0$

$\therefore \quad x = 0$ or $x = 1$

When $x = 0$, $y = 0 + 3 = 3$

$\therefore$ one point of intersection is $(0,3)$

When $x = 1$, $y = 1 + 3 = 4$

$\therefore$ other point of intersection is $(1,4)$

$\therefore$ Required area $= \int_0^1 \left[(x+3) - (x^2+3)\right] dx$

$$= \int_0^1 (x+3)dx - \int_0^1 (x^2+3)dx$$

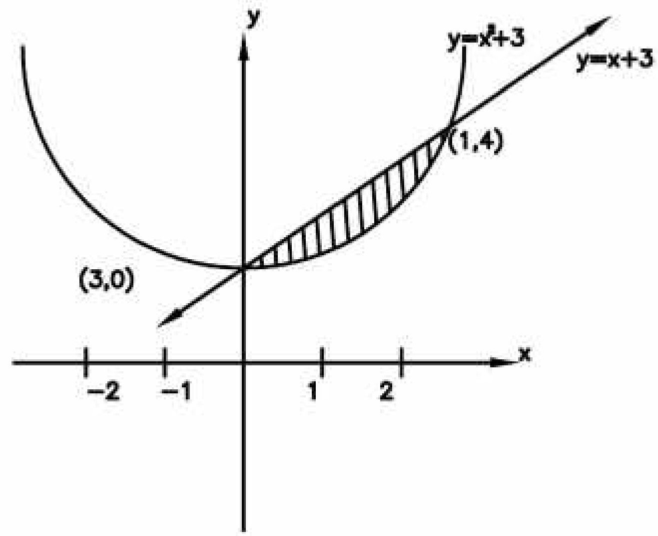
$$= \int_0^1 xdx + \int_0^1 3dx - \left[\int_0^1 x^2 dx + \int_0^1 3dx\right]$$

$$= \left[\frac{x^2}{2}\right]_0^1 + 3[x]_0^1 - \left[\frac{x^3}{3}\right]_0^1 - 3[x]_0^1$$

$$= \frac{1}{2}\left[1^2 - 0\right] + 3[1-0] - \frac{1}{3}\left[1^3 - 0\right] - 3[1-0]$$

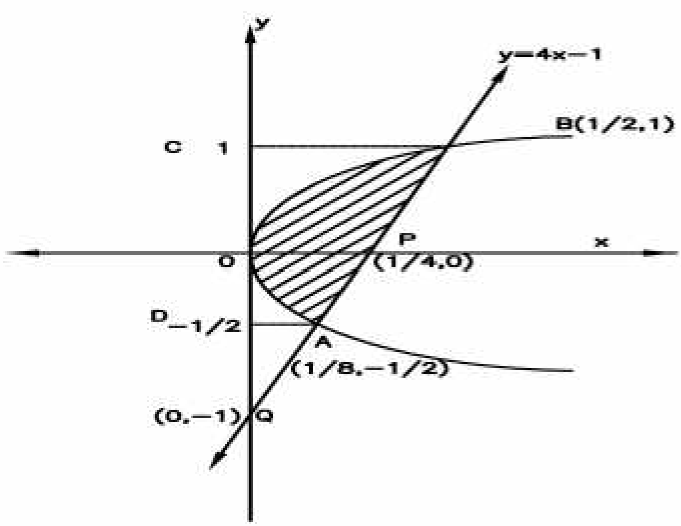
$$= \frac{1}{2} + 3 - \frac{1}{3} - 3$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$



Home work

Ex.21: Find the area of the bounded by the curve $y^2 = 2x$ and $y = 4x - 1$



3.Mean and RMS values.

With the help of Definite Integral Average or Mean value of the function $y = f(x)$ can be calculated. Therefore If $y = f(x)$ is integrable over the interval $a \le x \le b$ or $[a,b]$, then the mean value of the function $y = f(x)$ over $[a,b]$ is given by the formula,

$$\overline{Y} \text{ or } Y_{mean} \text{ or } Y_{avg} = \frac{1}{b-a}\int_a^b y dx = \frac{1}{b-a}\int_a^b f(x)dx$$

Note:-

1. Trignometric functions 'sinx' and 'cosx' are periodic with period 2® .
2. The period of 'sinpx' and 'cospx' is $T = \frac{2®}{P}$.
3. Therefore for period T of Function $y = f(x)$,

$$\overline{Y} \text{ or } Y_{mean} \text{ or } Y_{avg} = \frac{1}{T}\int_a^b y dx = \frac{1}{T}\int_a^b f(x)dx$$

Examples1: Find the mean value of the function $y = 4 - x^2$ over $[0,2]$.

Solution:

Given: $y = 4 - x^2$ over $[0,2]$ $\therefore$ a=0, b=2

The mean value of the function $y = f(x)$ over $[a,b]$ is given by,

$$Y_{mean} = \frac{1}{b-a}\int_a^b y dx$$

$$= \frac{1}{2-0}\int_0^2 4 - x^2 dx$$

$$= \frac{1}{2}[4\int_0^2 dx - \int_0^2 x^2 dx]$$

$$= \frac{1}{2}[4\int_0^2 dx - \int_0^2 x^2 dx]$$

$$= \frac{1}{2}\left[4x_0^2 - \frac{x^3}{3}\Big|_0^2\right]$$

$$= \frac{1}{2}\left[8 - \frac{8}{3}\right]$$

$$= \frac{8}{3}.$$

Examples2: Find the mean value of the function $y = x \cdot \sqrt{x^2+3}$ in the range over $0 \le x \le 1$.

Solution:

Here: $y = f(x) = x \cdot \sqrt{x^2+3}$, a=0, b=1

The mean value of the function $y = f(x)$ over the range $0 \le x \le 1$ is given by,

$$Y_{mean} = \frac{1}{b-a}\int_a^b y dx$$

$$= \frac{1}{1-0}\int_0^1 x \cdot \sqrt{x^2+3} dx$$

$$= \int_0^1 x \cdot \sqrt{x^2+3} dx$$

The integral is evaluated by the method of substitution.

Taking $x^2 + 3 = t$ $\quad \therefore\ 2xdx = dt$ or $x \cdot dx = \frac{dt}{2}$

When $x = 0$, $t = 0 + 3 = 3$

When $x = 1$, $t = 1 + 3 = 4$

Then, the above integral (1) becomes,

$$Y_{mean} = \frac{1}{b-a}\int_3^4 \sqrt{t} \cdot \frac{dt}{2}$$

Examples3: Find the mean value of the function $y = x^2 - 4x + 3$ between the points where it cut x-axis.
Solution:

The Curve $y = x^2 - 4x + 3$ cuts the x-axis in the points where $y = 0$ **.**putting $y = 0$ in $y = x^2 - 4x + 3$ we get,

$\therefore x^2 - 4x + 3 = 0$
Factorizing, we have, $(x-3)(x-1) = 0$
$\therefore$ $x = 3$ or $x = 1$.
$\therefore$ Two points on x-axis are: (1,0) and (3,0).

The mean value of $y = f(x)$ over the range $1 \le x \le 3$ is:
Then, the above integral (1) becomes,

$$Y_{mean} = \frac{1}{b-a}\int_a^b y.dx$$

$$= \frac{1}{3-1}\int_1^3 \left(x^2 - 4x + 3\right)dx$$

$$= \frac{1}{2}\left[\int_1^3 x^2.dx - 4\int_1^3 x.dx. + 3\int_1^3 dx\right]$$

Examples4: Find the mean value of the $I = 10\sin 100\text{®}t$ over a complete period.
Solution:
Given the function as $I = 10\sin 100\text{®}t$
Comparing with $\sin pt$, we have $p = 100\text{®}$

$\therefore$ Period of the function, $T = \frac{2®}{P} = \frac{2®}{100®} = \frac{1}{50}$

Then, the mean value of the function $y = f(x)$ having period T is given by,

$$Y_{mean} = \frac{1}{T}\int_0^T I.dt$$

$$= \frac{1}{\frac{1}{50}}\int_0^{1/50} 10.\sin(100®t).dt$$

Remark – The mean value of trigonometric functions over a complete period is zero.

Homework.

Examples5: An alternating current is given by $i = 20\sin 100t$. **Find the mean value of ‘** i^2 **’ over a complete period.**

Examples6: The instantaneous value of an alternating current in amperes is given by $i = 20\sin \mu t + \sin 3\mu t$. **Find the mean value of the current over the range** $i = 0$ **to** $i = \frac{\circledR}{\mu}$ **.**

ROOT MEAN SQUARE (R.M.S.) VALUE:

The R.M.S. value of the function $y = f(x)$ over $[a,b]$ is given by the formula,

$$Y_{r.m.s.} = \sqrt{\frac{1}{b-a}\int_a^b y^2 dx}$$

Note:-

1) The R.M.S. value is also called the **effective value.** Therefore $Y_{r.m.s.} = Y_{eff}$
2) The R.M.S. value is generally applied only to periodic functions.
3) The R.M.S. value of any sinusoidal waveform taken over an interval equal to one period is $\frac{1}{\sqrt{2}}$ times amplitude of the waveform.
4) Mean values and R.M.S. values are very Useful in calculating current, e.m.f……etc.

Example1: Find the R.M.S. value of the function $f(x) = x^2$ **over the interval** $1 \le x \le 3$.

Solution:

Given**,** $y = f(x) = x^2$ and interval $1 \le x \le 3$. $\therefore$ a=1, b=3.

The R.M.S. value of the function $y = f(x)$ over $[a,b]$ is given by the formula,

$$Y_{r.m.s.} = \sqrt{\frac{1}{b-a}\int_a^b y^2 dx} \qquad …….(1)$$

Where $$I = \int_1^3 y^2 dx = \int_1^3 \left(x^2\right)^2 dx$$

$$= \int_1^3 x^4 dx$$

$$= \left[\frac{x^5}{5}\right]_1^3$$

$$= \frac{242}{5}$$

Therefore , from (1) we have:

$$Y_{r.m.s.} = \sqrt{\frac{1}{3-1} \cdot \frac{242}{5}} = 4.92$$

Example 2: Find the R.M.S. value of the function $f(t) = \sin wt + \cos wt$ **over** $[0,1]$

Solution:

Given, $y = f(t) = \sin wt + \cos wt$ over $[0,1]$ $\therefore$ a=0, b=1.

Then, $Y_{r.m.s.} = \sqrt{\frac{1}{b-a}\int_a^b y^2 dt}$

Where $I = \int_a^b y^2 dt = \int_0^1 (\sin wt + \cos wt)^2 dt$

$$= \int_0^1 \left(\sin^2 wt + 2\sin wt.\cos wt + \cos^2 wt\right) dt$$

Note that $\sin^2 wt + \cos^2 wt = 1$ and $2\sin wt.\cos wt = \sin(2wt)$

$\therefore$ $$I = \int_0^1 (1 + \sin 2wt)\, dt$$

Example 3: Find the R.M.S. value of the function $I = 3\sin 2t$ **over a complete cycle.**

Solution:

Given : $I = 3\sin 2t$ over a complete cycle

$\therefore$ Period of I is $T = \frac{2\text{®}}{P}$ where p=2

Comparing $\sin 2t$ with $\sin pt$.

$\therefore T = \frac{2\text{®}}{2} = \text{®}$

Examples4: Find R.M.S. value of an alternating current $i = 5\sin 200\text{®}t$.

Solution:

Given $i = 5\sin 200\text{®}t$

Comparing $\sin 200\text{®}t$ with $\sin \text{®}t$, $\sin 200\text{®}t$

$\therefore$ Period of the function, $T = \frac{2\text{®}}{P} = \frac{2\text{®}}{200\text{®}} = \frac{1}{100}$

Then $i^2_{r.m.s.} = \frac{1}{T}\int_0^T i^2 .dt$

….Note that we are taking square of $i_{r.m.s.}$ to avoid root sign.

$$= \frac{1}{\frac{1}{100}} \int_{0}^{1/100} \{5 \sin 200®t\}^2 .dt$$

Examples5: An alternating current is given by $i = a \sin t$ **. Find the R.M.S value of the current over a half wave.**

Solution:

Given $i = a \sin t$ over a half wave

$\therefore$ The range of the function is $t = 0$ to $i = ®$ (half of $2®$)

$\therefore$ $a = 0$ to $b = ®$

HW.

Examples6: Find R.M.S. value of the function $y = a + b\cos x$ **over the interval** [0,®].

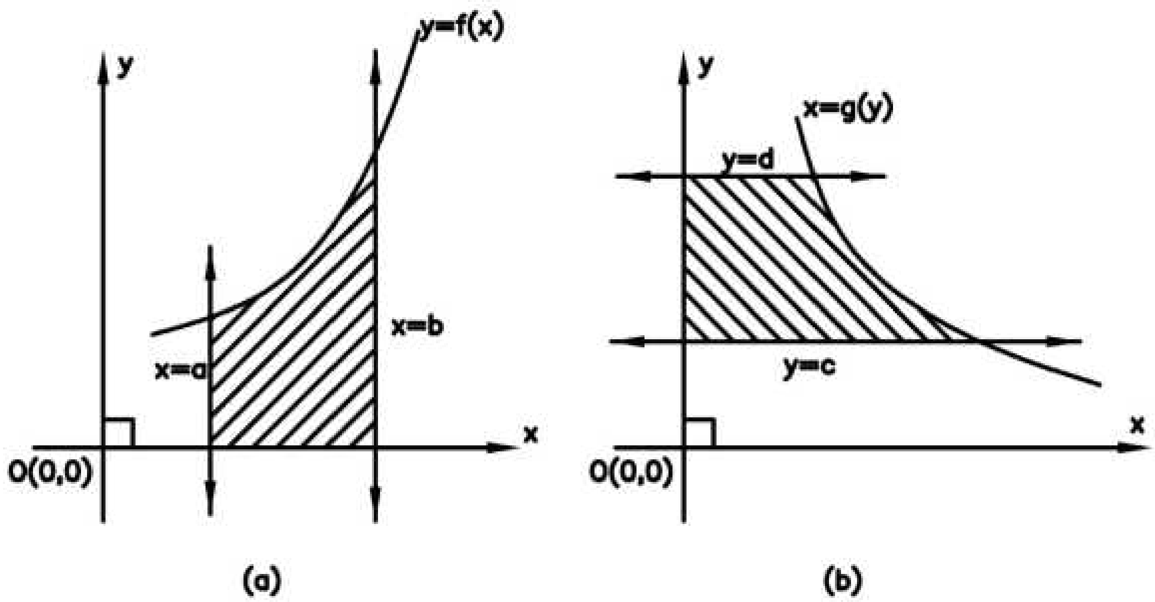
4.Volume of solid revolution:-

Consider $y = f(x)$ be a continuous function defied on the interval $[a,b]$. Fig a.
Then, the volume of the solid obtained by revolving the area under $y = f(x)$ from $x = a$ to $x = b$ with x-axis abount x-axis is given by the formula

$$V = \text{®}\int_a^b y^2.dx = \text{®}\int_a^b [f(x)]^2.dx$$

Similarly, the volume of the solid generated by revolving the area bounded by the curve $x = g(y)$, y-axis and lines $y = c$, $y = d$ abount y-axis is given by the formula:

$$V = \text{®}\int_{y=c}^{y=d} x^2.dy = \text{®}\int_c^d [g(y)]^2.dy \qquad \text{Refer Fig.}$$



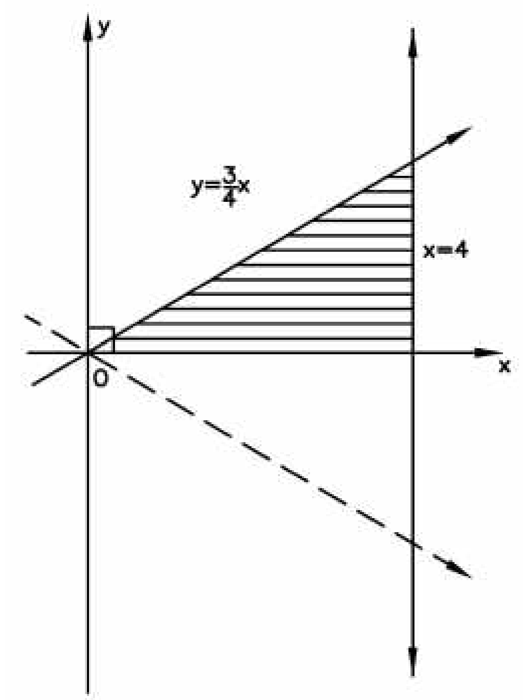
Note:

1. If a rectangle is revolved about one of its sides, we obtain a right circular cylinder as the solid of revolution.
2. If a right-angled triangle is revolved about one of its legs, we obtain a right circular cone as the solid of revolution.
3. If a semi-circle is revolved about its diameter, we obtain a sphere of the same radius as the solid of revolution.

Examples1: Find the volume of right circular cone generated by revolving the line $y = \frac{3}{4}x$ **about x-axis between the ordinates** $x = 0$ to $x = 4$

Solution:

The problem is represented diagrammatically as shown in fig.



When the line $y = \frac{3}{4}x$ is revolved about x-axis between the ordinates $x = 0$ to $x = 4$, the volume of solid cone so generated is given by,

$$V = \pi \int_0^4 y^2 .dx$$

$$= \pi \int_0^4 \left(\frac{3}{4}x \right)^2 .dx$$

$$= \frac{9\pi}{16} \int_0^4 x^2 .dx \quad = \quad = \frac{9\pi}{16} \frac{x^3}{3}$$

$$= \frac{3®}{16} \cdot \left[4^3 - 0\right] \quad = \quad = \frac{3®}{16} \times 64$$

$$= 3® \times 4 \quad = \quad 12® \text{ cubic units.}$$

Examples2: Find the volume of solid obtained by revolving about x-axis the plane area bounded by the curve $y = 2\sin 3x$, **x-axis and ordinates** $x = 0$ to $x = \frac{®}{3}$

Solution:

volume of solid of revolution is given by,

$$V = ® \int_a^b y^2 .dx$$

$$= ® \int_0^{®/3} \left(2\sin 3x\right)^2 .dx$$

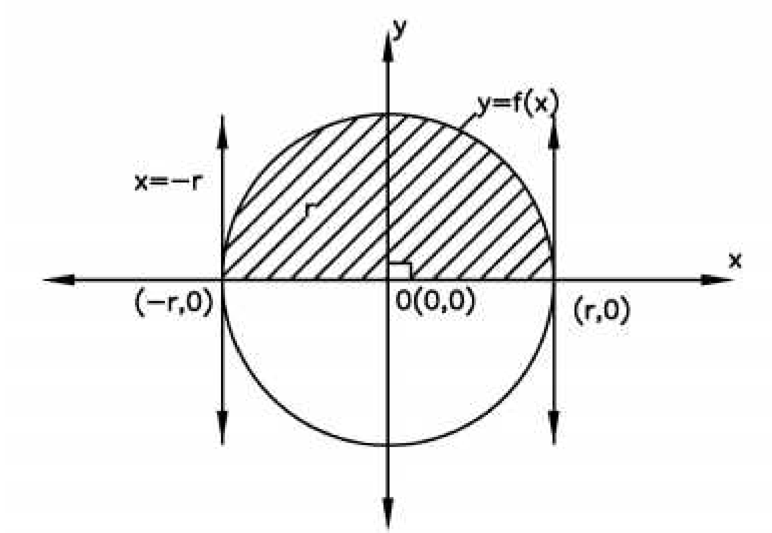
Examples3: Find the volume generated by revolving semi-circle abount its bounding diameter

OR

Find the volime of a sphere of radius r using integration.

Solution:

Consider a circle with centre at origin, that is, $O\left(0.0\right)$ and radius r, as shown in fig.



The equation of circle with centre at origin and radius r, is

$$x^2 + y^2 = r^2, \qquad y^2 = r^2 - x^3$$

The area of the semi-circle bounded by its diameter, that is, the area under $y = f(x)$ from $x = -r$ to $x = r$ with x-axis is when revolved about x-axis, a solid so obtained is a sphere of the same radius $(i.e. r)$. Its volume is given by,

$$V = ® \int_{-r}^{r} y^2 .dx$$

$$= ® \int_{-r}^{r} \left(r^2 - x^2\right) dx \qquad \text{..... From (1),} \quad y^2 = r^2 - x^3$$

$$= 2® \int_{0}^{r} \left(r^2 - x^2\right) dx \qquad \text{.....} \because f(x) = r^2 - x^2 \text{ is even}$$

By property of definite integral $\int_{-a}^{a} ......dx = 2\int_{0}^{a} ....dx$

$$= 2® \left[r^2 \int_0^r dx - \int_0^r x^2 dx \right]$$

$$= 2® \left[r^2 .x \Big|_0^r - \frac{x^3}{3} \Big|_0^r \right]$$

$$= 2® \left[r^2 .(r - 0) - \frac{1}{3}\left(r^3 - 0\right) \right]$$

$$= 2® \left[r^3 - \frac{r^3}{3} \right] \quad = 2® \left[\frac{3r^3 - r^3}{3} \right] = 2® . \frac{2r^3}{3}$$

$$= \frac{4}{3}®r^3 \text{Cubic units.}$$

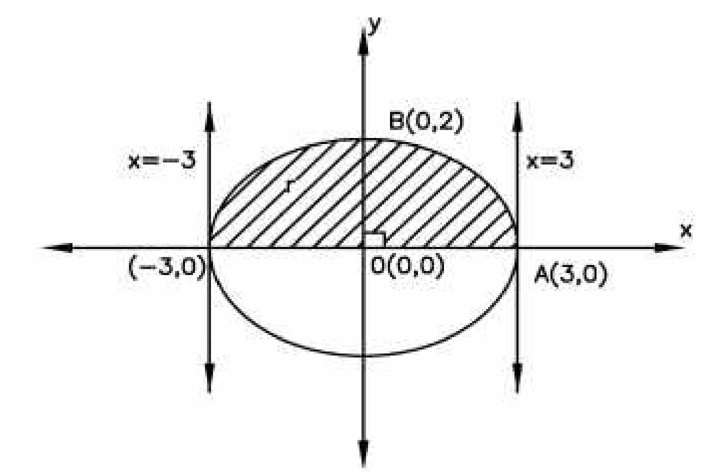
Examples4: Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **about the x-axis.**

Solution:

The equation of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Re-writing for y^2 , we get

$$\frac{y^2}{4} = 1 - \frac{x^2}{9} = \frac{9 - x^2}{9} = 1$$



Examples5: Find the volume obtained by revolving the area under the curve $9x^2 - 4y^2 = 36$ in the interval from $x = 2$ to $x = 4$ about x-axis.

Solution:

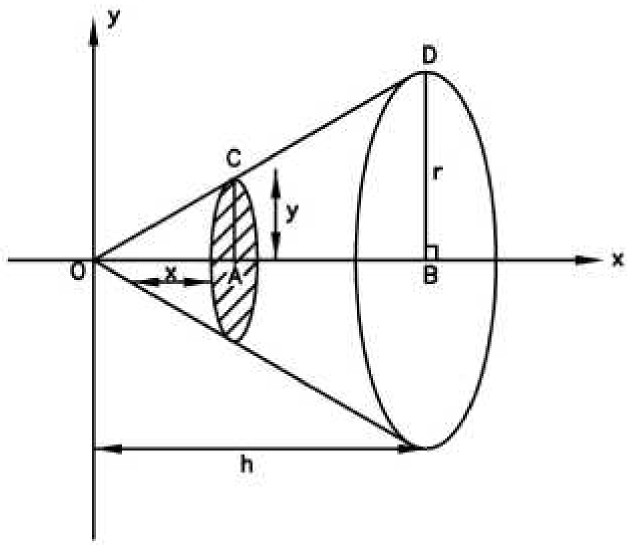
The equation of the curve is $9x^2 - 4y^2 = 36$ (which is a hyperbola)

$$9x^2 - 4y^2 = 36 \text{ or } y^2 = \frac{9}{4}\left(x^2 - 4\right)$$

Examples6: Find the formula for the volume of a right circular cone of height 'h' and base radius 'r' by using integration.

Solution:

In fig.



For two similar triangles their corresponding sides are in production.

$$\therefore \frac{y}{x} = \frac{r}{h}$$

$$\therefore y = \frac{r}{h} \cdot x$$

Now, the volume of right circular cone is given by

$$V = ® \int_0^h y^2 .dx$$

Examples7: Find the volume of the solid obtained by revolving the region bounded by the curve $y = x$ and $y = x^2$ **about x-axis.**

Solution:

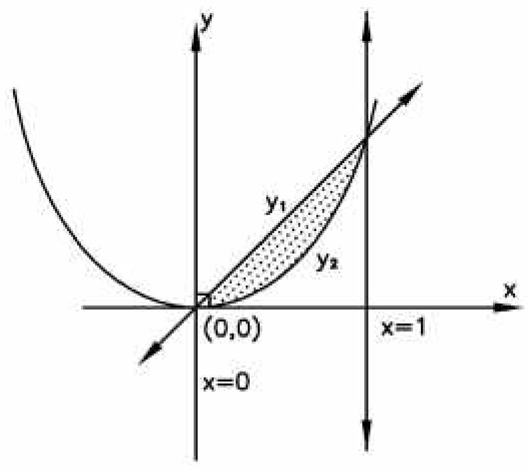
The point of intersection of the curves $y = x$ and $y = x^2$ are obtained equating (for y)them.

$\therefore \; x = x^2 \quad \therefore \; x^2 - x = 0 \quad \therefore \; x(x-1) = 0 \quad \therefore \; x = 0$ or $x = 1$

When $x = 0$ or $y = 0$ $\quad \therefore$ one point of intersection is $(0,0)$

When $x = 1$ or $y = 1$ $\quad \therefore$ one point of intersection is $(1,1)$

The area of revolution to get solid is as shown in fig.



Where $y_1 = x \quad \therefore \; y_1^2 = x^2$

$y_2 = x^2 \quad \therefore \; y_2^2 = x^4$

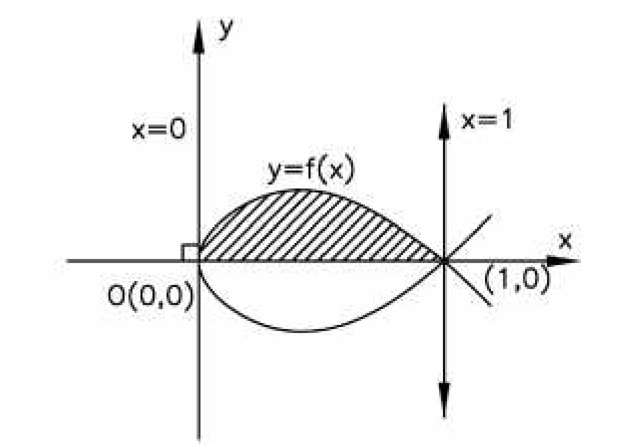
The required volume of the solid obtained by revolving the shaded area is given by,

$$V = ® \int_0^1 \left(y_1^2 - y_2^2\right) dx$$

Examples8: The loop of the curve $y^2 = x(x-1)^2$ is rotated about the x-axis. Find the volume of the solid so generated.

Solution:

The graph of the curve $y^2 = x(x-1)^2$ is as shown in the fig.



The graph intersects x-axis in the point where $y = 0$

$$\therefore \; 0 = x(x-1)^2 \quad \therefore \; x = 0 \text{ or } x = 1$$

Point of intersection are $(0,0)$ and $(1,0)$

The required volume of the solid generated by revolving the shaded area about x-axis is given by,

$$V = ®\int_0^1 y^2 .dx$$

$$= ®\int_0^1 x(x-1)^2 .dx$$